Modeling and Simulation of Interaction of Fluxons via CNN

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***Abstract*—This paper deals with interaction of fluxons. Usually fluxons arise in the well-known Josephson Junction (JJ) which is used in many applications in superconductor electronics, as for example in THz radiation. We shall study the modification of Sine-Gordon equation which is used for the investigation of fluxons. We shall model interaction of fluxons from the point of view of CNN. Multiple Sine-Gordon equations will be investigated. Extensive simulations will be presented.**

***Keywords*—fluxons,modified Sine-Gordon equation, multiple Sine-Gordon equations,modeling, simulation, interaction of fluxons, CNN**

I. INTRODUCTION

The physical object called fluxon is a quantum of magnetic flow. It is stable in the sense that it can be conserved, when its direction is changed and it can contact other electronic devices. The advantage of fluxons is that they process information with a very high speed and with a very low energy supply. That is why, fluxons have applications in information processing of electronic devices. When arising in Josephson Junction (JJ) [2] the equations which describe fluxons are the following:

$$I = I_0 \sin\varphi \tag{1}$$

$$\frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} v \tag{2}$$

where I presents the lossless supercurrent flowing through the JJ, I_0 is the maximum value which presents the current of JJ, the difference between the phases of the complex pair wave functions of the both superconductors is denoted by $\varphi(t) = \theta_1 - \theta_2$. The voltage drop over the junction is v and the fundaments constant known as single flux quantum is $\Phi_0 = 2.07\ mV.ps$. JJ circuit is given on Fig. 1:

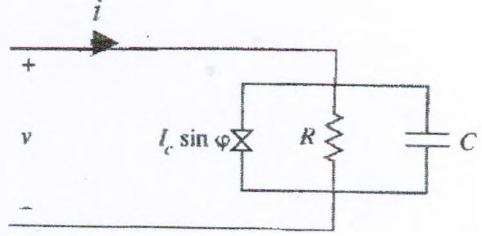

Fig.1. JJ circuit element.

In this paper we shall investigate the generalized form of equation (1), namely

$$I = I_0 \sin\varphi + [G_0(v) + G_1(v)\cos\varphi]v \tag{3}$$

G_0 and G_1 are rather complex functions of the voltage and the temperature. For the sake of simplicity we shall consider G_0 and G_1 to be constants. In this way equations (2) and (3) can be written in the following form:

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial t^2} - \alpha(1 + \varepsilon\cos\varphi)\frac{\partial\varphi}{\partial t} = \sin\varphi - \gamma \tag{4}$$

where γ, $\Gamma \equiv G_0(\Phi_0/2\pi I_0 C)^{\frac{1}{2}} = const > 0$, $0 < \varepsilon \equiv G_1/G_0 \ll 1$, $\alpha \in [10^{-2}, 10^{-4}]$, C is the capacitance. If we take α very small in equation (4) then it is transformed into the famous Sine-Gordon equation [6]. In the literature equation (4) describes the dissipation in JJ [2]. In our modeling in the next section we shall consider $G_0 = g_0(v)$ and $G_1 = g_1(v)$ to be of quadratic form.

In Section 2 we shall first define fluxons and antifluxons. Then we shall present the interaction of fluxons. In Section 3 CNN computing of interaction of fluxons will be provide through simulations. In Section 4 we shall consider CNN simulations of multiple Sine-Gordon equations.

II. MODELING OF INTERACTION OF FLUXONS

In this paper we shall study equation (4) in the case when $\varepsilon = 0$. Traveling wave solutions of (4) can be found in the form [2,3]:

$$\varphi = \sin^{-1}\gamma_0 + 2\sin^{-1}\left(cn\left(\frac{1}{k}\left(\frac{\gamma_0}{2\Gamma c^2}\right)^{\frac{1}{2}}(x-x_0),k\right)\right), \quad (5)$$

$k = \left(\frac{2\gamma_0}{\gamma+\gamma_0}\right)^{\frac{1}{2}}$, $\gamma_0 = \frac{2\Gamma c^2}{((1-c^2)^2+4\Gamma^2c^2)^{1/2}}$, c is the velocity of the wave.

The above solutions are describing geometrically by the loops varying from $-\infty$ to $+\infty$. When we have positive loop $\varphi \in (0,2\pi)$ and when we have negative loop $\varphi \in (-2\pi, 0)$. We shall use physical terminology [2], i.e. positive loops are called fluxons, and negative loops are called anti-fluxons.

In our modeling process we are interested in finding the solutions of modified Sine-Gordon equation (4) in the following form $\varphi = f(x)g(t)$, where $f(x) = sh\,\frac{ct}{\sqrt{1-c^2}}$, $0 < c^2 < 1$, $g(t) = \frac{1}{c\,ch\frac{x}{\sqrt{1-c^2}}}$. Our goal is to model the interaction of fluxons-anti-fluxons from physical point of view. Let us assume that the velocity $c \in (0,1)$. Having in mind that [3]

$$sh\,x \approx \begin{cases} -\frac{1}{2}e^{-x}, x \to -\infty \\ \frac{1}{2}e^{x}, x \to \infty \end{cases},$$

we obtain:

$$\varphi = \frac{1}{c}\begin{cases} -e^{\frac{x-ct}{\sqrt{1-c^2}}}, x \to -\infty \quad (IV) \\ e^{-\frac{x-ct}{\sqrt{1-c^2}}}, x \to \infty \quad (II). \end{cases}$$

These solutions are anti-fluxons and they are monotonically decreasing [3] (see Fig.1):

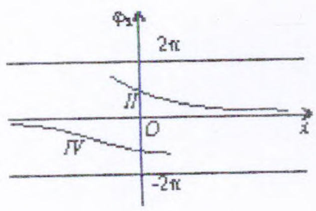

Fig.1. Anti-fluxons.

We obtain in a similar way [3]:

$$\varphi = \frac{1}{c}\begin{cases} e^{\frac{x+ct}{\sqrt{1-c^2}}}, x \to -\infty \quad (I) \\ -e^{-\frac{x+ct}{\sqrt{1-c^2}}}, x \to \infty \quad (III). \end{cases}$$

These solutions are fluxons and they are monotonically decreasing (see Fig.2):

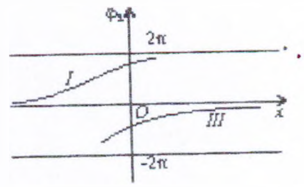

Fig.2. Fluxons.

It is obvious that, there is a shift: $x \to x - \sqrt{1-c^2}\ln\frac{1}{c}$. So, the interaction of fluxon-anti-fluxon [3] is given on Fig.3.

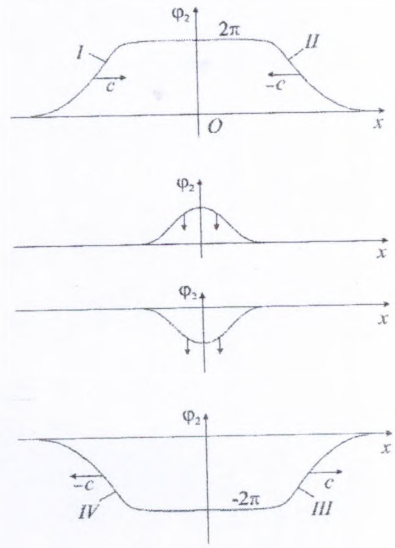

Fig.3. Interaction fluxon-anti-fluxon.

We shall now discuss the interaction of pair of fluxons with velocities c_1 and c_2. It is given on Fig.4 [3]:

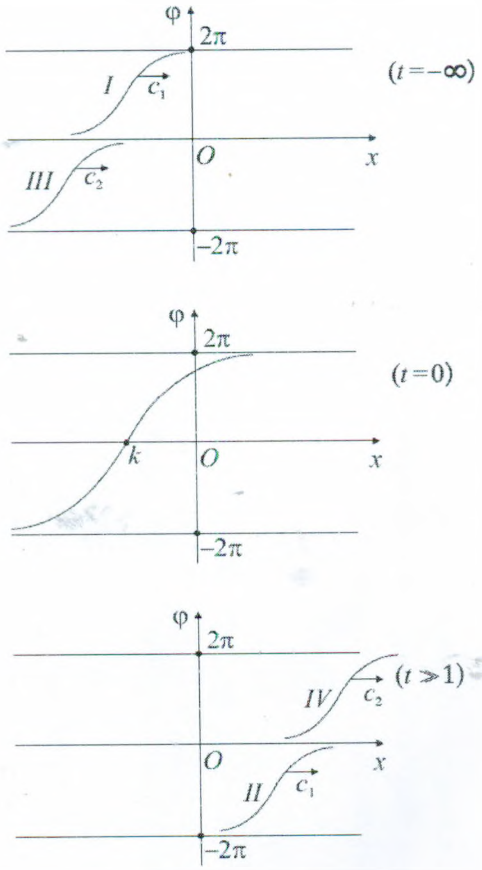

Fig.4. Interaction of two fluxons.

We suppose that $c_2 > c_1 > 0$. The fluxon I has the phase x_1 and the velocity c_1. Fluxon III is with phase y_1 and the same velocity c_1. After the collision with the second fluxon, the first one gets negative shift. Therefore, the slower fluxon moving to the right with velocity $c_1 > 0$ is shifting additionally backward. In a similar way we obtain the interaction of fluxons IV and II. But in this case after the collision the second fluxon gets the positive shift, and therefore faster fluxon moving to the right with velocity $c_2 > c_1 > 0$ is shifting additionally forward.

Remark 1. The slower (*I*) and the faster (*III*) fluxons are starting for $t = -\infty$, (*III*) being located to the left with respect to I and $I \in \{\varphi > 0\}$, $III \in \{\varphi < 0\}$. At $t = 0$ the fluxon I is joining III and they are forming the configuration V [3]. We have $\varphi(0, x) \to 2\pi$ in the case $x \to +\infty$ and $\varphi(0, x) \to -2\pi$ in the case $x \to -\infty$. When $t \ll 1$ the two fluxons are regenerating with the same velocities c_1, c_2 and the same profiles. They are denoted by IV (the faster one) and by II (the slower one). The fluxon IV is located to the right with respect to II, $IV \in \{\varphi > 0\}$ and $II \in \{\varphi < 0\}$.

There is one other wave profile which is called in the literature [2] "breathon". This a solution of (4) of the following type [3] $\varphi = 4\tan^{-1}\frac{\tan x \sin[(\cos\sigma)t]}{\cosh[(\sin\sigma)t]}$, σ is a parameter and $|\sigma| < \frac{\pi}{2}$. On Fig. 5 we present the simulation of such wave:

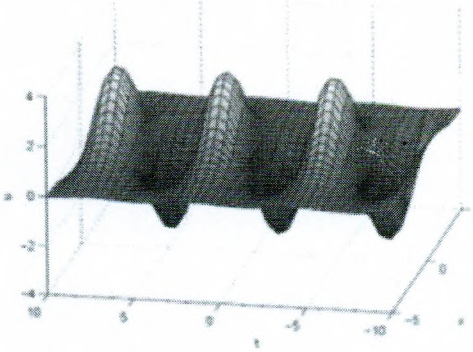

Fig.5. "Breathon" wave.

III. CNN Simulations of Interaction of Fluxons

In [2] R.Parmentier shows that the modified Sine-Gordon equation (4) has two different from physical point of view solutions - plasma waves corresponding to the swing (oscillation) of the pendulum and fluxon waves corresponding to the rotation of the pendulum.

In the previous section we studied the interaction of fluxons from the theoretical point of view. In this section we shall present CNN computing of interaction of fluxons. By using CNN cell dynamics [1] we shall discretize equation (4) in the following way. We map $\varphi(x, t)$ into a CNN layer such that the state voltage of a CNN cell at a grid point is v_j. Let us consider one-dimensional CNN, where the CNN cells consist of a linear capacitor in parallel with a nonlinear inductor described by $i_j = f(v_j) = \alpha(1 + \varepsilon\cos v_j)u_j - \sin v_j - \gamma$ and where these cells are coupled to each other by linear inductors with inductance L.

In terms of CNN circuit topology we can identified the following corresponding elements [4]:

1). CNN cell dynamics:

$$\frac{du_j}{dt} = \frac{1}{C}\left[I_j - f(v_j)\right], \qquad (6)$$

$$\frac{dv_j}{dt} = u_j, 1 \le j \le N;$$

2). CNN synaptic law:

$$I_j = i_{L_j} - i_{L_{j+1}} = \frac{1}{L}(v_{j-1} - 2v_j + v_{j+1}) \qquad (7)$$

where $v_j(t) = \int_{-\infty}^{t} u_j(\tau)d\tau$ is the flux-linkage at node j. Here the synaptic law (7) is in fact a discrete Laplacian $A = [1, -2, 1]$ of the flux linkage v_j.

Based on the equations (6) and (7) CNN simulation of fluxon is given of Fig.6.

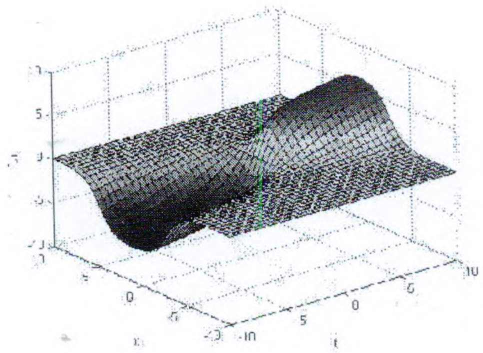

Fig.6. CNN computing of fluxon.

IV. CNN MODELING OF DIFFERENT TYPES OF SINE-GORDON EQUATIONS

In the literature different types of Sine-Gordon equation are known [5]. In this section we shall present modeling of so called multiple Sine-Gordon equations. As first example we consider the following double Sine-Gordon equation:

$$\varphi_{xx} - \varphi_{tt} = \pm(\sin\varphi + \frac{1}{2}\lambda\sin\frac{1}{2}\varphi) \qquad (8)$$

When $\lambda = 1$ equation (8) is unstable [5]. Such type of unstable equations describe the spread of excitation of amplifier. Simulation of the interaction of solutions of type "kink" of (8) with negative sign and $\lambda = 1$is given on Fig. 7:

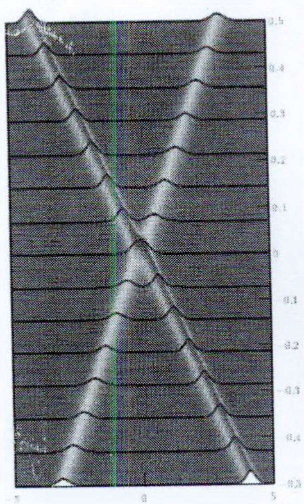

Fig.7. 4π "kink" waves interaction.

As a second example we shall consider triple Sine-Gordon equation:

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi + \frac{1}{3}\sin\frac{1}{3}\varphi + \frac{2}{3}\sin\frac{2}{3}\varphi \qquad (9)$$

Simulations of solutions of (9) is given on Fig.8.

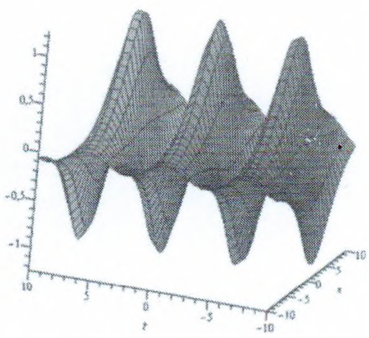

Fig.8. Solutions of triple Sine-Gordon equation (9).

Let us consider now the interaction of $(4\pi - 2\delta)$ kink-antikink solution of (8). In this case some energy may be lost and then long living "breathon" appears. The equation which describes such phenomena is [5,6]:

$$\varphi_{xx} - \varphi_{tt} = -\left(\sin\varphi + \frac{1}{2}\sin\frac{1}{2}\varphi\right) - \gamma\omega\Omega_l^{-2}B_1\sin(\omega\Omega_l^{-1}t)$$

We shall derive next CNN model of multiple Sine-Gordon equations [4].

1). CNN cell dynamics:

$$\frac{du_j}{dt} = \frac{1}{C}[I_j - g(v_j)] \qquad (10)$$

$$\frac{dv_j}{dt} = u_j, 1 \le j \le N;$$

$$g(v_j) = -\left(\sin v_j + \frac{1}{2}\sin\frac{1}{2}v_j\right) - \gamma\omega\Omega_l^{-2}B_1\sin(\omega\Omega_l^{-1}t)$$

2). CNN synaptic law:

$$I_j = i_{L_j} - i_{L_{j+1}} = \frac{1}{L}(v_{j-1} - 2v_j + v_{j+1}). \qquad (11)$$

For this CNN model we can apply double Fourier transform [4]:

$$F(s,z) = \sum_{k=-\infty}^{k=\infty} z^{-k}\int_{-\infty}^{\infty} f_k(t)e^{-st}\,dt\,, \qquad (12)$$

$s = i\omega$, and $z = e^{i\Omega}$, ω being a temporal frequency, and Ω being a spatial frequency.

After applying describing function method [4] we obtain the following result when the array is circular with finite set of spatial frequencies:

$$\Omega = \frac{2\pi K}{T}, 0 \le K \le N - 1, \qquad (12)$$

T is a minimal period.

Proposition 1. CNN model (10), (11) of circular array of N identical, inductively coupled Josephson Junctions has periodic state solution $(v_j(t), u_j(t))$ with a finite set of spatial frequencies (12).

Remark 2. In order to obtain good characterization of the periodic steady state solutions of CNN model (10), (11) we should have initial conditions such that the network to be able to reach a steady state solution for the values of Ω form (12). That is why in our simulations below we take the following initial conditions: $v_j = \sin \Omega j, j = 1,2, \cdots, N$.

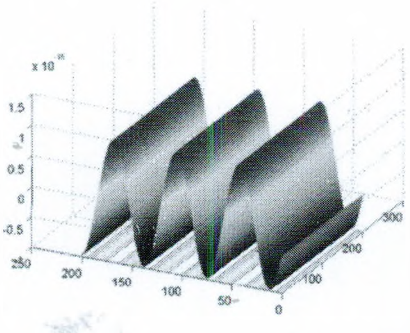

Fig.9. Periodic solutions of CNN model (10), (11).

V. CONCLUSIONS

In this paper we study different types of Sine-Gordon equations and their solutions. We define fluxons from both mathematical and physical point of view. We model the interaction of fluxon-anti-fluxon, two fluxons, 4π "kink" type solutions. CNN computing is presented in order to do the simulations of interaction of fluxons, and multiple Sine-Gordon equations. We show the relation between Josephson Junctions and fluxons, as well as the relation between JJ and CNN.

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