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Abstract. In this paper Harmonic balance technique (HBT) is presented for studying Cellular Nonlinear Network (CNN) model of Meinhardt-Gierer equation (CNN-MG). First, short survey on HBT and CNN is given. Then we prove existence of periodic solutions of CNN-MG model. Computer simulations illustrate the obtained theoretical results.

INTRODUCTION

The harmonic balance technique is a very powerful method for predicting the dynamic behavior of periodic steady state solution of system differential equations. It is very easy to implement in computer systems. Many aspects of qualitative behavior have to be investigated numerically. For this purpose we apply the Cellular Nonlinear Networks (CNN) approach for studying such models.

Harmonic balance technique for studying differential equations

The nonlinear system of differential equations can be presented as the system called Lur'e system /see fig 1/.

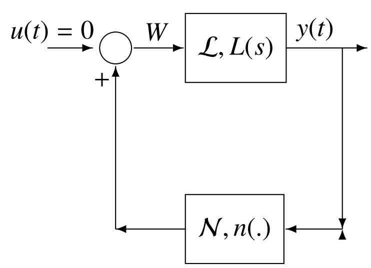

FIGURE 1. Lur'e scheme.

The block $\mathcal{L}$ is presented by transfer function into frequency domain. The signal entering through the nonlinear block of the Lur'e scheme [5] is approximated by means of a suitable sinusoidal term whose frequency and amplitude are unknown. The higher-order harmonics in the output of the nonlinear block are neglected. We shall consider a basic Lur'e scheme [5], with $\mathcal{L}$ - linear time - invariant dynamic system and $\mathcal{N}$ - nonlinear time-invariant static and memoryless system.

This means that the output signal of the system is

$$y(t) = (-n(y(t)).L(s) \tag{1}$$

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Which is equal to the so - called harmonic balance equation.

$$y(t) + (n(y(t)).L(s) = 0 \tag{2}$$

We assume that nonlinearity is then represented by the Fourier series as $n[y_0(t)] = N_0(A, B) + N_1 B \cos(\omega t) + ...$, where

$$N_0 = \frac{1}{2\pi A} \int_{-\pi}^{\pi} n[y_0(t)] d\omega t, \tag{3}$$

$$N_1 = \frac{1}{2B} \int_{-\pi}^{\pi} n[y_0(t)] d\omega t. \tag{4}$$

Equilibrium points can be obtained through

$$n^{'}(E_j) = \frac{dn(y)}{dy}|_{y=E_j}. \tag{5}$$

The prediction of boundary cycles is made by the conditions:

$$1 + N_0(A, B)L(0) = 0 \tag{6}$$

$$1 + N_1(A, B)L(j\omega) = 0 \tag{7}$$

which are described in a system as parameters A, B and ω.
The condition $1 + N_1(A, B)L(j\omega) = 0$ graphically corresponds to the intersection of the Nyquist plot $L(j\omega)$ with the function $-\frac{1}{N_1(A,B)}$.

Cellular Nonlinear Network

One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear.

General CNN which cells are made of time-invariant circuit elements are arranged in arrays of cells (Fig.2).

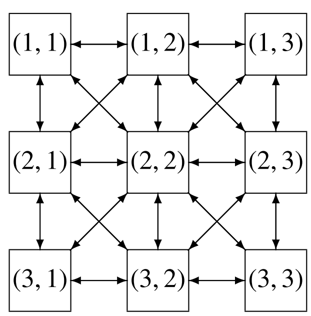

FIGURE 2. 3×3 neighborhood CNN.

Each cell $C(ij)$ have CNN cell dynamics, CNN synaptic law which represents the interactions (spatial coupling) within the neighbor cells; Mathematically cell dynamic is described with:

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I^s_{ij}), \tag{8}$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbor cell $C(i + k, j + l)$ are specified by a CNN synaptic law:

$$\begin{aligned} I^s_{ij} &= A_{ij,kl} x_{i+k,j+l} + \\ &+ \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \\ &+ \tilde{B}_{ij,kl} * u_{i+k,j+l}(t). \end{aligned} \tag{9}$$

The first term $A_{ij,kl}x_{i+k,j+l}$ of (9) is simply a linear feedback of the states of the neighboring nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

Algoritm for studying dynamic CNN via harmonic balance technique

Let us consider the following system:

$$\begin{aligned}&\dot{x}_i(t) = -x_i(t) + sy_{i-1}(t) + py_i(t) + sy_{i+1}(t)\\&y_i = f(x_i(t)),\\&1 \leq i \leq N,\end{aligned} \tag{10}$$

where $f(.)$ is piecewise linear function and s$\geq \frac{p-1}{2}$
We look for periodic solutions of the form:

$$x_i = \xi(\Omega_0 j + \omega_0 t) \tag{11}$$

Where $\xi : R \rightarrow$R and some $0 \leq \Omega_0 \leq 2\pi$, $\omega_0 = \frac{2\pi}{T_0}$

Let's apply continuous time discrete Fourier transform from continuous time t and discrete space j into discrete space frequency Ω and discrete time frequency ω:

$$\tilde{X}_\Omega(\omega) = \tilde{X}_k(\omega) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} x_j(t) e^{-j\frac{2k\pi}{N}+\omega t} dt \tag{12}$$

When we apply this transformation to (10), we receive the transfer function

$$H(\omega_0, \Omega_0) = \frac{X(\omega_0, \Omega_0)}{Y(\omega_0, \Omega_0)} = \frac{se^{-j\Omega_0} + p + se^{j\Omega_0}}{1 + j\omega_0} \tag{13}$$

After some transformation we get:

$$H(\omega_0, \Omega_0) = \frac{p + 2ssin\Omega_0}{1 + \omega_0^2} + j\frac{2ssin\Omega_0 - p\omega_0}{1 + \omega_0^2} \tag{14}$$

For the real and imaginary part of the transfer function we obtain:

$$\begin{aligned}Re[H(\omega_0, \Omega_0)] &= \frac{p+2ssin\Omega_0}{1+\omega_0^2} = \frac{X_{m_0}}{Y_{m_0}}\\Im[H(\omega_0, \Omega_0)] &= \frac{2ssin\Omega_0 - p\omega_0}{1+\omega_0^2} = 0\end{aligned} \tag{15}$$

We suppose that the state variable and output variable have the form

$$\begin{aligned}x_j(t) &= X_{m_0} sin(\omega_0 t + j\Omega_0)\\y_j(t) &= Y_{m_0} sin(\omega_0 t + j\Omega_0),\end{aligned} \tag{16}$$

Where the amplitude X_{m_o},temporal frequency ω_0 and spatial frequency Ω_0 are therefore the unknowns to be determined.

We approximate the periodic output $y_i(t) = f(x_i(t))$ by the fundamental component of its Fourier series and calculate Y_{m_0} by formula:

$$Y_{m_0} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(X_{m_0} sin(\psi)) sin(\psi) d\psi \tag{17}$$

From equations (15) we obtain the unknowns

$$\begin{aligned}\omega_0 &= \frac{2s}{p} sin\Omega_0\\X_m &= \frac{2p}{\pi}[X_m arcsin\frac{1}{X_m} + \sqrt{1 - \frac{1}{X_m^2}}]\end{aligned} \tag{18}$$

System (10) with $s > \frac{p-1}{2}$, possess at least $\frac{n-1}{2}$ different nontrivial periodic solutions, whose spatial frequencies are $\Omega_0 = \frac{2\pi k}{N}; 1 \leq N \leq \frac{n-1}{2}$

APPLICATION OF HARMONIC BALANCE METHOD TO MEINHARDT-GIERER MODEL

In this section we shall apply the HBT to Meinhardt-Gierer equation. The described model is used to model the regeneration of the head of hydra or of multicellular organisms composed of several cell types. Its dynamics is given by the following nonlinear system of differential equations:

$$\begin{aligned} \frac{\partial u(t,x,y)}{\partial t} &= au^2(t,x,y)\frac{1}{v(t,x,y)} - \beta u(t,x,y) + D_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial v(t,x,y)}{\partial t} &= au^2(t,x,y) - \gamma v(t,x,y) + D_2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial v^2}{\partial y^2}\right) \end{aligned} \tag{19}$$

First equation describes the change of activator concentration, where u(t,x,y) a is a short-range auto catalytic substance, i.e., activator,
v(t,x,y) is its long-range antagonist, i.e., inhibitor.
$\frac{\partial u}{\partial t}$ describes the change of activator concentration u(t,x,y) per time unit.
u^2 describes the production rate of activator which depends in a non-linear way on it concentration .
$\frac{1}{v}$ Shows the fact that the activator production is captured by the inhibitor.
β is a coefficient, indicating the degree of inhibition of the activator molecules. The exchange of molecules takes place through diffusion,γ is is an inhibitor loading factor.

We map solutions u(t,x,y) and v(t,x,y) of equation (19) into a CNN layer such that a stage voltage of CNN cell $x_{kl}(t)$ in a grid point (k,l) is associated with $u(t, kh_x, lh_y)$ $h_x = \triangle x, h_y = \triangle y$. We assume that $h_x = h_y = h$. Then the expression of second derivative is

$$u_{xx}+u_{yy} \sim \frac{1}{h^2}((u(t,x+h,y)-u(t,x,y)-(u(t,x,y)-u(t,x-h,y))+(u(t,x,y+h)-u(t,x,y)-(u(t,x,y)-u(t,x,y-h))) \tag{20}$$

it can be written such that

$$u_{xx} + u_{yy} \sim \frac{1}{h^2}(u_{k+1,l} - 4u_{k,l} + u_{k,l-1} + u_{k-1,l} + u_{k,l-1}) \tag{21}$$

Then we can write down cells dynamic of (19) as:

$$\begin{aligned} \dot{u}_{i,j} &= \alpha\frac{u_{i,j}^2}{v_{i,j}} - \beta u_{i,j} + D_1(u_{i-1,j} + u_{i,j+1} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) \\ \dot{v}_{i,j} &= \alpha u_{i,j}^2 - \gamma v_{i,j} + D_2(v_{i-1,j} + v_{i,j+1} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j}) \end{aligned} \tag{22}$$

We apply the Fourier transform into (22):

$$F(s, z_1, z_2) = \sum_{k=-\infty}^{k=\infty} z_1^{-k} \sum_{l=-\infty}^{l=\infty} z_2^{-l} \int_{-\infty}^{\infty} f_{kl}(t)e^{-st}dt \tag{23}$$

Then for Meinhardt - Gierer model we obtain:

$$\begin{aligned} sU &= \alpha\frac{U^2}{V} - \beta U + D_1(z_1^{-1} + z_2^{-1} + z_1^1 + z_2^1 - 4)U \\ sV &= \alpha U^2 - \gamma V + D_2(z_1^{-1} + z_2^{-1} + z_1^1 + z_2^1 - 4)U \end{aligned} \tag{24}$$

After some transformations we get:

$$\begin{aligned} U &= \frac{1}{(s+\beta-D_1T)}N_1(U,V) \\ V &= \frac{1}{(s+\gamma-D_2T)}N_2(U,V) \end{aligned} \tag{25}$$

Where $T = z_1^{-1} + z_1 + z_2^{-1} + z_2 - 4 = 2\cos\Omega_1 + 2\cos\Omega_2 - 4$, $N_1(U,V) = \alpha\frac{U^2}{V}$, $N_2(U,V) = \alpha U^2$

$$\frac{U}{V} = \frac{s+\gamma-D_2T}{s+\beta-D_1T}N \tag{26}$$

when $N = \frac{N_1(U,V)}{N_2(U,V)}$.

This mean, that Meinhardt - Gierer system differential equations can be represented in the Lure scheme with transfer function $H(s, z_1, z_2) = \dfrac{s + \gamma - D_2T}{s + \beta - D_1T}$. We want to find solutions in the form :

$$\begin{aligned} u_{ij}(\omega_0, \Omega_1, \Omega_2) &= U_{m_0} \sin(\omega_0 t + k\Omega_1 + l\Omega_2) = U_{m_0} \sin(\psi) \\ v_{ij}(\omega_0, \Omega_1, \Omega_2) &= V_{m_0} \sin(\omega_0 t + k\Omega_1 + l\Omega_2) = V_{m_0} \sin(\psi) \end{aligned} \tag{27}$$

where $\psi = \omega_0 t + k\Omega_1 + l\Omega_2$, $\omega_0 = \frac{2\pi}{T_0}$, where $T_0 > 0$ is the minimal period. We take periodic boundary conditions for our CNN model (22) and we suppose that $s = j\omega_0$, $z_2 = e^{j\Omega_2}$ and $z_1 = e^{j\Omega_1}$, where ω_0 is a temporal frequency, Ω_1, Ω_2 are the spatial frequencies, and $\Omega_1 + \Omega_2 = \frac{2k\pi}{n}, where 0 < k \leq n - 1$. We replace s, z_1 an z_2 in (26)

Then we obtain:

$$H(\omega_0, \Omega_1, \Omega_2) = \frac{-\omega_0^2 - (\gamma - D_2T)(\beta - D_1T)}{-\omega_0^2 - (\beta - D_1T)^2} + j\frac{(\gamma - D_2T) - (\beta - D_1T))}{-\omega_0^2 - (\beta - D_1T)^2}\omega_0 \tag{28}$$

Transfer function is real function. Then

$$\begin{aligned} Re(H(\omega_0, \Omega_1, \Omega_2) &= \tfrac{-\omega_0^2 - (\gamma - D_2T)(\beta - D_1T)}{-\omega_0^2 - (\beta - D_1T)^2} = \tfrac{U_{m_0}}{V_{m_0}} \\ Im(H(\omega_0, \Omega_1, \Omega_2)) &= \tfrac{(\gamma - D_2T) - (\beta - D_1T))}{-\omega_0^2 - (\beta - D_1T)^2}\omega_0 = 0 \end{aligned} \tag{29}$$

According to harmonic balance technique we assume that corresponding nonlinearity N_{ij} is expanded in Fourier series as:

$N_{ij} = N_0(U_{m_0})U_{m_0} + N_1(U_{m_0})U_{m_0} \sin\psi + ...$

Coefficients are calculating as follow:

$$\begin{aligned} N_0(U_{m_0}) &= \tfrac{1}{2\pi U_{m_0}} \int_{-\pi}^{\pi} N(U_{m_0} \sin\psi) d\psi = 0 \\ N_1(U_{m_0}) &= \tfrac{1}{\pi U_{m_0}} \int_{-\pi}^{\pi} N(U_{m_0} \sin\psi) \sin\psi d\psi = \tfrac{2}{U_{m_0} V_{m_0}} \end{aligned} \tag{30}$$

Then according with [2] and harmonic balance technique existence of periodic solution of (22) hold for the next equations:

$$\begin{aligned} &U_{m_0}[1 + N_0(U_{m_0})H(0, \Omega_1, \Omega_2)] = 0 \\ &1 + N_1(U_{m_0})H(i\omega_0) = 0 \end{aligned} \tag{31}$$

Then we obtain:

$$\gamma - D_2T - (\beta - D_1T) = 0 \tag{32}$$

Then:

$$\begin{aligned} &T = \tfrac{\gamma - \beta}{D_2 - D_1} \\ &T = e^{-j\Omega_1} + e^{j\Omega_1} + e^{-j\Omega_2} + e^{j\Omega_2} - 4 \\ &T = 2\cos\Omega_1 + 2\cos\Omega_2 - 4 \\ &2\cos\Omega_1 + 2\cos\Omega_2 - 4 = \tfrac{\gamma - \beta}{D_2 - D_1} \\ &2\cos\tfrac{k\pi}{n}\cos(\tfrac{2\Omega_1 - \frac{2k\pi}{n}}{2}) = \tfrac{\gamma - \beta}{D_2 - D_1} \\ &\cos(\Omega_1 - \tfrac{k\pi}{n}) = \tfrac{\gamma - \beta + 8(D_2 - D_1)}{4\cos\frac{k\pi}{n}(D_2 - D_1)} \\ &\Omega_1 = \tfrac{k\pi}{n} + \arccos\tfrac{\gamma - \beta + 8(D_2 - D_1)}{4\cos\frac{k\pi}{n}(D_2 - D_1)} \\ &\Omega_2 = \tfrac{k\pi}{n} - \arccos\tfrac{\gamma - \beta + 8(D_2 - D_1)}{4\cos\frac{k\pi}{n}(D_2 - D_1)} \end{aligned} \tag{33}$$

We can approximate output $\frac{U}{V}$ of the Lure system with

$$\frac{U}{V} \sim \frac{U_{m_0}}{V_{m_0}} \sin(\omega_0 t + k\Omega_1 + l\Omega_2) \tag{34}$$

$$\frac{U_{m_0}}{V_{m_0}} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(U_{m_0}\sin\psi)\sin\psi d\psi = \frac{2}{V_{m_0}} \tag{35}$$

where $f(x) = \frac{1}{2}(|x-1|-|x+1|)$. When we calculate (35) we obtain that $U_{m_0} = 2$

According the harmonic balance technique we obtain

$$\begin{aligned} &Re(H(\omega_0, \Omega_1, \Omega_2) = \frac{-\omega_0^2-(\gamma-D_2T)(\beta-D_1T)}{-\omega_0^2-(\beta-D_1T)^2} = \frac{U_{m_0}}{(V_{m_0})} = \frac{2}{V_{m_0}} \\ &V_{m_0} = \frac{(\omega_0^2+(\gamma-D_2(2\cos\Omega_1+2\cos\Omega_2-4))^2)}{2(\omega_0^2+(\gamma-D_2(2\cos\Omega_1+2\cos\Omega_2-4))(\beta-D_1(2\cos\Omega_1+2\cos\Omega_2-4)))} \\ &U_{m_0} = 2 \end{aligned} \tag{36}$$

By using the harmonique balance method we've proven this theorem:

Theorem 1 *Meinhardt - Gierer model (22) with circular array with* $n = M \times M$ *cells has periodic solution* $u_j(t)$ *and* $v_j(t)$ *with period* $T_0 = \frac{2\pi}{\omega_0}$ *and amplitude* U_{m_0} *for all* $\Omega_0 + \Omega_2 = \frac{2\pi k}{n}, 0 \le k \le n-1$

RESULTS AND CONCLUSION

MathLab program was developed to analyze the dynamic behavior of the solution of the equations (19) of the Meinhardt - Gierer model which makes it easy to visualize the graphical results obtained.

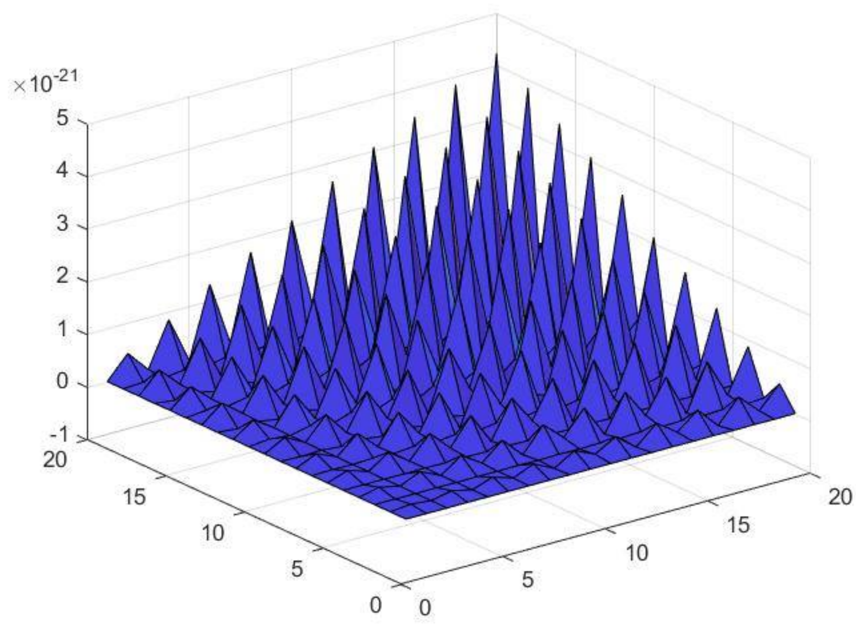

FIGURE 3. simulation of function v of meinhardt-gierer model

REFERENCES

[1] L.O. Chua and L. Yang, *Cellular Neural Network: Theory and Applications*, IEEE Trans. CAS. vol. 35, p.1257, 1988.

[2] R.Genesio and A.Tesi, *A harmonic balance approach for chaos prediction: the Chua's circuit*, Int.J.of Bifurcation and Chaos, 2:61-79, 1992.

[3] R.Genesio and A. Tesi, *H armonic balance methods for the analysis of chaotic dynamics in nonlinear systems*, Automatica, 28:531-548, 1992.

[4] A.I. Mees, *Dynamics of Feedback Systems*, London, Wiley, 1981.

[5] A.Slavova, *Cellular Neural Networks:Dynamics and Modeling*, Kluwer Academic Publishers, 2003.

[6] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Philadelphia:Society for Industrial and Applied Mathematics, 2002

[7] Hans Meinhardt *http://www.scholarpedia.org/article/Gierer-Meinhardt˙model*,Hans Meinhardt (2006) Gierer-Meinhardt model. Scholarpedia, 1(12):1418.