Optimization of Linear Objective Function Under Max-product Fuzzy Relational Constraint

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Abstract: This paper presents optimization problem with a linear objective function subject to constraint – fuzzy linear system of equations or fuzzy linear system of inequalities, with max-product composition. Methods that provide algorithm for computing the maximum solution and all minimal solutions, when the fuzzy linear system is consistent, are developed for constraint resolution. A software is developed in MATLAB and Java.

Key words: Linear optimization, fuzzy relational equations, fuzzy relational inequalities, max-product composition, inverse problem resolution

1 Introduction

The main problem that is solved here is to optimize (minimize or maximize) the linear objective function

$$Z = \sum_{j=1}^{n} c_j x_j, c_j \in \mathbb{R}, 0 \leq x_j \leq 1, 1 \leq j \leq n, \quad (1)$$

with traditional addition and multiplication, subject to fuzzy linear system of equations (inequalities, respectively) as constraint

$$A \odot X = B \;\; (A \odot X \geq B, \text{respectively}), \qquad (2)$$

where $A = (a_{ij})_{m\times n}$ stands for the matrix of coefficients, $X = (x_j)_{n\times 1}$ stands for the matrix of unknowns, $B = (b_i)_{m\times 1}$ is the right-hand side of the system and for each $i, 1 \leq i \leq m$, $j, 1 \leq j \leq n$, a_{ij}, b_i, $x_j \in [\,0,\,1\,]$. The composition written as $\odot$ is max-product, $c = (c_1, ..., c_n)$ is the weight (cost) vector. The results for solving this optimization problem are provided by the inverse problem resolution for fuzzy linear systems as presented in [11], [14] for equations and next developed here. Solving fuzzy linear systems with max-product composition is subject of great scientific interest. The main results are published in [1], [2], [4],[10], [13], [14] for equations.

Since the feasible domain of the solution set is non-convex, traditional linear programming methods, such as simplex method, can not be applied. We propose to obtain complete solution set of the system, using algebraic-logical approach. Then the optimization problem is spitted into two sub-problems by separating the non-negative from negative coefficients of the objective function, such that the optimal solution is obtained by the maximum solution and one of the minimal solutions of the system.

2 Basic Notions

A matrix $A = (a_{ij})_{m\times n}$, with $a_{ij} \in [\,0,\,1\,]$ for each i, $j, 1 \leq i \leq m$, $1 \leq j \leq n$, is called a *membership matrix* [6]. In what follows "matrix" is used, instead of "membership matrix".

For $X = (x_j)_{n\times 1}$ and $Y = (y_j)_{n\times 1}$ the inequality $X \leq Y$ means $x_j \leq y_j$ for each j, $1 \leq j \leq n$.

Partial order relation on a partially ordered set (poset) P is denoted by the symbol $\leq$. *Greatest element* of a poset P is an element $b \in P$ such that $x \leq b$ for all $x \in P$. The *least element* of P is defined dually.

Product algebra:

$$\mathbb{I}_{\odot} = \langle [\,0,\,1\,],\, \vee,\, \wedge,\, 0,\, 1,\, \odot \rangle \qquad (3)$$

with operations:

$$a \vee b = max\{a, b\} \qquad (4)$$

$$a \wedge b = min\{a, b\} \qquad (5)$$

and $\odot$ – traditional multiplication, is used here. In $\mathbb{I}_{\odot}$ the operation $\diamond$ is defined as follows:

$$a \diamond b = \begin{cases} 1, & \text{if } a \le b \\ \frac{b}{a}, & \text{if } a > b \end{cases} . \quad (6)$$

Any fuzzy relation $R \in F(X \times Y)$ over finite support $X \times Y$ is representable by a matrix [4], written for convenience with the same letter $R = (r_{ij})$, where $r_{ij} = \mu_R(x_i, y_j)$ for any $(x_i, y_j) \in X \times Y$. In what follows, the matrix $R = (r_{ij})$ is used instead of the fuzzy relation $R \in F(X \times Y)$ over finite support $X \times Y$.

For each j, $j = 1, \cdots, n$,

$$A^*(j) = (a^*_{ij})_{m \times 1} \quad (7)$$

denotes the $j-$th column of A^* and a^*_{ij} denotes the $i-$th element ($1 \le i \le m$) in $A^*(j)$.

Definition 1 *Let the matrices $A = (a_{ij})_{m \times p}$ and $B = (b_{ij})_{p \times n}$ be given.*

- The matrix $C = (c_{ij})_{m \times n} = A \odot B$ is called $max-$*product* of A and B if

$$c_{ij} = \max_{k=1}^{p}(a_{ik}.b_{kj}), 1 \le i \le m, 1 \le j \le n. \quad (8)$$

- The matrix $C = (c_{ij})_{m \times n} = A \diamond B$ is called $min-$ $\diamond$ of A and B if

$$c_{ij} = \min_{k=1}^{p}(a_{ik} \diamond b_{kj}), 1 \le i \le m, 1 \le j \le n. \quad (9)$$

Theorem 2 *Let $A = (a_{ij})_{m \times p}$ and $C = (c_{ij})_{m \times n}$ be given matrices and let $\mathbb{B}_{\odot}$ be the set of all matrices such that $A \odot B = C$. Then:*

- *$\mathbb{B}_{\odot} \ne \emptyset$ iff $A^t \diamond C \in \mathbb{B}_{\odot}$.*
- *If $\mathbb{B}_{\odot} \ne \emptyset$ then $A^t \diamond C$ is the greatest element in $\mathbb{B}_{\odot}$. [3]*

3 Solving Fuzzy Linear Systems

Let first consider the solution set of fuzzy linear system of equations (FLSE) with $\max -product$ composition:

$$\left| \begin{array}{ccccc} (a_{11}.x_1) & \vee \cdots \vee & (a_{1n}.x_n) & = & b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a_{m1}.x_1) & \vee \cdots \vee & (a_{mn}.x_n) & = & b_m \end{array} \right. , \quad (10)$$

written in the following equivalent matrix form

$$A \odot X = B, \quad (11)$$

or fuzzy linear systems of inequalities (FLSI) with the same left-hand side as (10) written in the following equivalent matrix form

$$A \odot X \ge B. \quad (12)$$

When the results or statements are valid for (11) and (12) simply FLS is used instead of listing them.

Definition 3 *Let the FLS in n unknowns be given.*

- *$X^0 = (x^0_j)_{n \times 1}$ with $x^0_j \in [0, 1]$, when $1 \le j \le n$, is called a solution of FLS if $A \odot X^0 = B$ ($A \odot X^0 \ge B$, respectively) holds.*
- *The set of all solutions $\mathbb{X}^0$ of FLS is called complete solution set.*
- *A solution $X^0_{\text{low}} \in \mathbb{X}^0$ is called a lower (minimal) solution of FLS if for any $X^0 \in \mathbb{X}^0$ the relation $X^0 \le X^0_{\text{low}}$ implies $X^0 = X^0_{\text{low}}$. Dually, a solution $X^0_{\text{u}} \in \mathbb{X}^0$ is called an upper (maximal) solution of FLS if for any $X^0 \in \mathbb{X}$ the relation $X^0_{\text{u}} \le X^0$ implies $X^0 = X^0_{\text{u}}$. When the upper solution is unique, it is called greatest (or maximum) solution.*
- *If $\mathbb{X}^0 \ne \emptyset$ then FLS is called consistent, otherwise it is called inconsistent.*

Any consistent system has unique maximum solution

$$X_{gr} = A^t \diamond B \quad (13)$$

for (11) or

$$X_{gr} = (1, ..., 1)^t \quad (14)$$

for (12).

The solution set of FLS is determined by all minimal solutions and the maximum one.

3.1 Associated matrix

For the FLS any coefficient $a_{ij} \ge b_i$ provides a way to satisfy the $i-$th equation (inequality, respectively) with $a_{ij} . x_j = b_i$, when $x_j = \frac{b_i}{a_{ij}}$. A symbolic matrix $A^* = (a^*_{ij})$ with elements a^*_{ij} determined according to the next expression:

$$a^*_{ij} = \begin{cases} S, & \text{if } a_{ij} < b_i \\ E, & \text{if } a_{ij} = b_i \\ G & \text{if } a_{ij} > b_i \end{cases} . \quad (15)$$

is assigned to FLS in order to distinguish coefficients that contribute for solving the system from these that do not contribute.

The matrix A^* is called *associated matrix* of the FLS.

3.2 IND vector

The vector $IND = IND_{m\times 1}$ is used to establish consistency of the system. $|G_i|$ is the number of elements $a^*_{ij} = G$ and $|E_i|$ is the number of elements $a^*_{ij} = E$ in the $i-$th row of A^*, $j = 1, ..., n$. Then

$$IND(i) = |G_i| + |E_i|. \tag{16}$$

Lemma 4 *Let the FLS be given. Then:*

- *If $IND(i) = 0$ for at least one $i = 1, ..., m$ then the system is inconsistent.*
- *If the system is consistent then the number of its potential minimal solutions does not exceed*

$$PN = \prod_{i=1}^{m} IND(i). \tag{17}$$

The time complexity function for establishing the IND vector is O(mn).

3.3 Main theoretical results

Theorem 5 *Let the system (11) (12, respectively) be given.*

- *If $A^*(j)$ contains $G-$type coefficient(s) $a^*_{ij} = G$ and*

  $$\hat{x}_j = \min_{i=1}^{m} \{\frac{b_i}{a_{ij}}\}(\hat{x}_j = 1, \textit{respectively}),$$

  when $a_{ij} > b_i$, then $x_j = \hat{x}_j$ implies:

  - *$a_{ij} \,.\, x_j = b_i$ in (11) ($a_{ij} \,.\, x_j \geq b_i$ in (12)) for each i, $1 \leq i \leq m$ when $\frac{b_i}{a_{ij}} = \hat{x}_j$ ($\frac{b_i}{a_{ij}} \leq 1$, respectively),*
  - *$a_{ij} \,.\, x_j < b_i$ for each i, $1 \leq i \leq m$ with $\frac{b_i}{a_{ij}} \neq \hat{x}_j$ ($\frac{b_i}{a_{ij}} > 1$, respectively).*
- *If $A^*(j)$ does not contain any $G-$type coefficient, but it contains $E-$type coefficient(s) $a^*_{kj} = E$, then $\hat{x}_j = 1$ and $x_j = \hat{x}_j = 1$ implies $a_{ij} \,.\, x_j = b_i$ for each i, $1 \leq i \leq m$ with $a^*_{ij} = E$,*
- *If $A^*(j)$ contains neither $G-$ nor $E-$type coefficient then $\hat{x}_j = 1$ and $x_j = \hat{x}_j = 1$ implies $a_{ij} \,.\, x_j < b_i$ for each i, $1 \leq i \leq m$.*

The proof for Theorem 5 follow from the definition of the associated matrix and its relationship with the corresponding FLS.

Corollary 6 *For any consistent system (11) $X_{\mathrm{gr}} = A^t \diamond B = \hat{X} = (\hat{x}_j)_{n\times 1}$. For any consistent system (12) $X_{\mathrm{gr}} = (1, \ ..., \ 1)^t$. Here $\hat{x}_j$, $1 \leq j \leq n$, are computed according to Theorem 5.*

Corollary 7 *If $a^*_{ij} = S$ for each $i = 1, ..., m$, then $\check{x}_j = 0$ in any minimal solution $\check{X} = (\check{x}_j)_{n\times 1}$ of the consistent FLS.*

Corollary 8 *If $\check{X} = (\check{x}_j)_{n\times 1}$ is a minimal solution of the consistent FLS, then for each $j = 1, ..., n$ either $\check{x}_j = 0$ or $\check{x}_j = \hat{x}_j$.*

3.4 Selected elements

Definition 9 *For FLS with associated matrix A^* all non-S elements in A^* are called selected.*

The following corollary is obtained from Theorem 5.

Corollary 10 *Let a FLS be given.*

- *If it is consistent, then for each i, $1 \leq i \leq m$, there exists at least one selected coefficient a^*_{ij}.*
- *If the system is consistent then $X_{\mathrm{gr}} = A^t \diamond B$ is the unique maximal (i.e. greatest, or maximum) solution of (11) and $X_{\mathrm{gr}} = (1, \ ..., \ 1)^t$ is the unique maximal (i.e. greatest, or maximum) solution of (12).*
- *The time complexity function for establishing the consistency of the FLS and for computing X_{gr} is $\mathrm{O(m^2n)}$.*

3.5 Finding minimal solutions

For a consistent FLS all equations (inequalities) with $b_i = 0$ are removed.

3.5.1 Algebraic properties

If the element $a^*_{ij} \neq S$, then $m^*_{ij} = \frac{b_i}{a_{ij}}$ and this is symbolized this with $\left\langle \frac{m^*_{ij}}{j} \right\rangle$.

For each i, $1 \leq i \leq m$, the elements $m^*_{ij} \neq 0$ mark the potential lower bounds of different ways to satisfy the i-th equation (inequality) of the FLS, written E_i and symbolized by the sign $\sum$:

$$E_i = \sum_{1\leq j\leq n} \left\langle \frac{m^*_{ij}}{j} \right\rangle. \tag{18}$$

Equations are considered simultaneously. The *concatenation* W of all ways is symbolized by the sign $\prod$:

$$W = \prod_{1\leq i\leq m} \left(\sum_{1\leq j\leq n} \left\langle \frac{m^*_{ij}}{j} \right\rangle \right). \tag{19}$$

Concatenation is *distributive* with respect to addition, i.e.

$$\left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left(\left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle + \left\langle \frac{m^*_{i_2j}}{j_3} \right\rangle \right) =$$

$$= \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle + \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_3} \right\rangle . \quad (20)$$

Concatenation is *commutative*:

$$\left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle = \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle . \quad (21)$$

The next property is called *absorption for multiplication*:

$$\left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_1} \right\rangle = \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \quad (22)$$

Applying (20), (21), (22) the parentheses in (19) can be expanded. The set of ways follows, from which the minimal solutions are obtained:

$$W = \sum_{(j_1,\cdots,j_m)} \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle ... \left\langle \frac{m^*_{i_mj}}{j_m} \right\rangle . \quad (23)$$

Let simplify (23) according to the next described *absorption for addition* (missing $\left\langle \frac{m_{ij}}{j} \right\rangle$ are supposed to be $\left\langle \frac{0}{j} \right\rangle$):

$$\left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \cdots \left\langle \frac{m^*_{i_mj}}{j_m} \right\rangle + \left\langle \frac{m^*_{s_1j}}{j_1} \right\rangle \cdots \left\langle \frac{m^*_{s_mj}}{j_m} \right\rangle =$$

$$= \begin{cases} \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \cdots \left\langle \frac{m^*_{i_mj}}{j_m} \right\rangle, \\ \text{if } m^*_{i_tj} \le m^*_{s_tj} \text{ for } t = 1, \cdots, m \\ \\ \text{unchanged, otherwise} \end{cases} . \quad (24)$$

A property called *combined absorption* follows from (22), (23) and (24):

$$\left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left[\left\langle \frac{m^*_{i_2j}}{j_1} \right\rangle + \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle \right] =$$

$$= \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_1} \right\rangle + \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle =$$

$$= \left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle . \quad (25)$$

After simplifying (23) according to (24) any term $\left\langle \frac{m^*_{i_1j}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j}}{j_2} \right\rangle \cdots \left\langle \frac{m^*_{i_mj}}{j_m} \right\rangle$ determines a minimal solution $\check{X} = (\check{x}_j)$, with components (obtained after expanding brackets in (19) by rules (20) – (25)), see also Corollary 8:

$$\check{x}_{j_t} = \begin{cases} m_{i_tj} = \hat{x}_{j_t} \text{ if in (23) } m_{i_tj} \neq 0 \\ 0 \qquad\qquad \text{otherwise} \end{cases} \quad (26)$$

4 Algorithm for finding optimal solution

The linear objective function $Z = (c_1, c_2, ..., c_n)$ is decomposed in two functions $Z' = (c'_1, c'_2, ..., c'_n)$ and $Z'' = (c''_1, c''_2, ..., c''_n)$ by separating the non-negative and non-positive coefficients (as it is proposed in [5], [7], [8], [9]). The components of Z' arenon-negative, the components of Z'' are non-positive.

The original problem: to minimize (maximize, respectively) Z subject of constraint (2) splits into two problems, namely to minimize (maximize, respectively) both

$$Z' = \sum_{j=1}^{n} c'_j x_j \quad (27)$$

and

$$Z'' = \sum_{j=1}^{n} c''_j x_j \quad (28)$$

with constraint (2), i.e. for the problem (1) Z takes its minimum $Z_{\min}$ (maximum $Z_{\max}$, respectively) when both Z' and Z'' take minimum (maximum, respectively).

Since the components c'_j, $1 \le j \le n$, in Z' are non-negative, Z' takes its minimum (maximum, respectively) among the minimal solutions (for the greatest solution, respectively) of (2). Hence for the problem (27) the optimal solution $\check{X} = (\check{x}_1, ..., \check{x}_n)$ is among the minimal solutions (for the greatest solution, respectively) of the system (2).

Since the components c''_j, $1 \le j \le n$, in Z'' are non-positive, Z'' takes its minimum (maximum, respectively) for the greatest solution (among minimal solutions, respectively) of (2). Hence for the problem (28) the optimal solution is for the greatest solution (among minimal solutions, respectively) of the system (2).

The optimal solution of the problem (1) with constraint (2) is $X^* = (x^*_1, ..., x^*_n)$, where

$$x^*_i = \begin{cases} \hat{x}_i, & \text{if } c_i < 0 \\ \check{x}_i, & \text{if } c_i \ge 0 \end{cases} \quad (29)$$

when computing $Z_{\min}$ and the minimal value is

$$Z^* = \sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} c_j'' \hat{x}_j + c_j' \check{x}_j. \qquad (30)$$

When computing $Z_{\max}$ the maximal value is

$$Z^* = \sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} c_j' \hat{x}_j + c_j'' \check{x}_j \qquad (31)$$

with

$$x_j^* = \begin{cases} \check{x}_j, & \text{if } c_j < 0 \\ \hat{x}_j, & \text{if } c_j \geq 0 \end{cases}. \qquad (32)$$

5 Software

Software is developed, based on the described algorithm. This software is divided into two independent programs, one written in Java and one written in MATLAB. Both have their advantages. Java program has its own GIU, which make it stand alone application for optimization. In addition it is faster than the MATLAB program. Of course because it is open source everybody can extend it or just use it in other Java applications. MATLAB program on the other hand doesn't have GIU, but can be run easy from the command line interpreter (CLI) of MATLAB. In this way the program can be used separately as well as in combination of any of the MATLAB packages for rapid producing of feature rich programs. In the next section some examples are given. All of them are tested in both Java and MATLAB application. The results are given like a snapshot of a MATLAB session but, of course, they are identical in the Java program.

6 Experimental results

Example: Minimize

$$Z = 5x_1 - 4x_2 + 8x_3 + 2x_4 - 3x_5 + 7x_6 \qquad (33)$$

subject to

$$A \odot X \geq B, 0 \leq x_j \leq 1, 1 \leq j \leq 6 \qquad (34)$$

where

$$A = \begin{pmatrix} 0.00 & 0.20 & 0.05 & 0.00 & 0.40 & 0.00 \\ 0.10 & 0.60 & 0.30 & 0.00 & 0.20 & 0.20 \\ 0.80 & 0.48 & 0.24 & 0.48 & 0.00 & 0.00 \\ 0.30 & 0.00 & 0.00 & 0.40 & 0.80 & 0.15 \\ 0.00 & 0.00 & 0.12 & 0.20 & 0.48 & 0.10 \\ 0.50 & 0.30 & 0.00 & 0.10 & 0.60 & 0.00 \end{pmatrix}, \qquad (35)$$

$$B' = \begin{pmatrix} 0.10 & 0.30 & 0.24 & 0.20 & 0.12 & 0.15 \end{pmatrix}, \qquad (36)$$

In MATLAB:

```
>>sol=fuzzy_maximize_mp(A,B,C)
sol =
    exists: 1
       low: [6x3 double]
  sol_numb: 3
       Ind: [5x1 double]
       hlp: [6x6 double]
         A: [6x6 double]
         B: [6x1 double]
         d: [5x6 double]
        gr: [1 1 1 1 1]
   final_x: [6x1 double]
   final_z: -7
         z: [2 -7 1]
```

The system of constraints has one greatest and 3 minimal solutions. They are listed below:

```
>>sol.gr
ans =
  1.0000
  1.0000
  1.0000
  1.0000
  1.0000

>>sol.low
ans =
       0        0        0
  0.5000   0.5000        0
  1.0000        0   1.0000
  0.5000        0        0
       0   0.2500   0.2500
       0        0        0
```

$$Z' = -4x_2 - 3x_5 \qquad (37)$$

$$Z'' = 5x_1 + 8x_3 + 2x_4 + 7x_6 \qquad (38)$$

The second and the fifth component in the optimal solution are taken from the greatest solution. The other components must be chosen from one of the minimal solutions.

Values of objective function for all minimal solutions:

```
>> sol.z
ans =
  2  -7   1
```

Minimum of the objective function is gained for the second minimal solution, hence

```
>> sol.final_z
ans =
  -7

>> sol.final_x
```

```
ans =
   0
   1
   0
   0
   1
   0
```

The execution time for this example is 0.060 seconds tested on computer with Pentium 4 processor on 1.8GHz, with 512MB RAM and MATLAB R2006b. The same example tested on computer with Intel Core 2 Duo processor on 1.86GHz, with 2GB RAM and Java 6.0 has execution time of 0.015 seconds.

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