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Software for Testing Linear Dependence in Fuzzy Algebra

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***Abstract**: We present efficient algorithms and functions in MATLAB workspace for testing linear dependence and linear independence in the fuzzy algebra. The computational complexity of all algorithms is polynomial. Testing results with comparison for execution time of the algorithms are given.*
***Keywords:** Fuzzy algebra, linear dependence.*

INTRODUCTION

In this paper we investigate linear dependence and linear independence, and propose several algorithms and software in MATLAB workspace for establishing these properties in polynomial time. We work in the fuzzy algebra $([0,1], \max, \min)$, where $[0,1] \subset R$ is the real closed unit interval with the order $\leq$ and the operations are taking maxima (max) and taking minima (min) between elements [4]. As in the classical linear algebra [7] these operations are conventionally extended to fuzzy vectors and fuzzy matrices. But linear algebra and fuzzy algebra are completely different, that requires special investigations for linear dependence over fuzzy vectors and fuzzy matrices [3, 4, 11].

The terminology for fuzzy sets is according to [6], for computational complexity - according to [1, 5].

1.BASIC NOTIONS

The ***fuzzy algebra*** [4] $([0,1], \max, \min)$ is an algebraic structure, where $[0,1] \subset R$ is the unit real closed interval, the operations are $\max = \vee$ and $\min = \wedge$. It means that for arbitrary $a, b \in [0,1]$, according to the order $\leq$ in $[0, 1]$ we have:

$$a \vee b = \max\{a,b\}, \qquad a \wedge b = \min\{a,b\}.$$

In $([0, 1], \max, \min)$ the operation α is defined [6] as follows:

$$a \alpha b = \begin{cases} 1, & a \leq b \\ b, & a > b \end{cases}.$$

A matrix $A_{m\times n} = (a_{ij})$, with elements $a_{ij} \in [0, 1]$ is called [6] ***fuzzy membership matrix***.

Let $A = (a_{ij})_{m\times p}$ and $B = (b_{ij})_{p\times n}$ be given fuzzy membership matrices.

i) $C_{m\times n} = (c_{ij}) = A \bullet B$ is called ***max-min*** product of A and B if

$$c_{ij} = \max_{k=1}^{p}(\min(a_{ik}, b_{kj})), \text{ when } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

ii) $C_{m\times n} = (c_{ij}) = A \alpha B$ is called α***-product*** of A and B if

$$c_{ij} = \min_{k=1}^{p}(a_{ik} \alpha b_{kj}), \text{ when } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

Theorem 1 [12] *Let* $A=(a_{ij})_{m\times p}$ *and* $C=(c_{ij})_{m\times n}$ *be given fuzzy membership matrices. Let* $\mathbf{B}_\bullet$ *be the set of all matrices such that* $A\bullet B=C$. *Then:*

i) $\mathbf{B}_\bullet \neq \varnothing$ *iff* $A^t \alpha C \in B_\bullet$.

ii) *If* $\mathbf{B}_\bullet \neq \varnothing$, *then* $A^t \alpha C$ *is the greatest element in* $\mathbf{B}_\bullet$.

2.SYSTEMS OF LINEAR EQUATIONS OVER FUZZY ALGEBRA

A vector v with m components $v_i \in [0,1]$, $i=1,...,m$, is called ***fuzzy vector***. We denote by V_m the set of all fuzzy vectors with m components.

An expression of the form

$$(\lambda_1 \wedge v_1)\vee(\lambda_2 \wedge v_2)\vee\ldots\vee(\lambda_n \wedge v_n),$$

where $v_i \in V_m$, $i=1,...,n$ and $\lambda_1,\ldots,\lambda_n \in [0,1]$, is called $\max-\min$ ***linear combination*** of the fuzzy vectors $v_i \in V_m$, $i=1,...,n$ with coefficients $\lambda_1,\ldots,\lambda_n$.

We can compute max-min linear combination of fuzzy vectors [10].

Definition 1. The fuzzy vector $B\in V_m$ is called ***max-min linear combination*** of the fuzzy vectors $A=\{a_1,\ldots a_n\}\subseteq V_m$ if there exist $X\in V_n$ such that $A\bullet X=B$.

In order to test linear dependence we implement solving fuzzy linear systems of equations that have the following matrix form:

(1) $$A\bullet X=B$$

where $A=(a_{ij})_{m\times n}$ stands for the matrix of coefficients, $X=(x_j)_{n\times 1}$ stands for the matrix of unknowns, $B=(b_i)_{m\times 1}$ is the right-hand side of the system. For each i, $1\le i\le m$ and for each j, $1\le j\le n$, we have a_{ij}, b_i, $x_j\in[0,1]$ and the $\max-\min$ composition is written shortly as $\bullet$. To test whether a fuzzy vector $B\in V_m$ is a max-min linear combination of the fuzzy vectors $A=\{a_1,\ldots a_n\}\subseteq V_m$ we have to solve the system $A\bullet X=B$ for the unknown fuzzy vector $X\in V_n$. The next two theorems are in the core of the algorithms for testing linear dependence.

Theorem 2. [9] *It is algorithmically decidable in polynomial time whether the fuzzy linear system* $A\bullet X=B$ *is consistent or inconsistent*.

Theorem 3. *If the system* $A\bullet X=B$ *is consistent, its maximum solution* $X_{gr}=A^t\alpha B$ *has components* $X_{gr}(j)=\min_i\{b_i : a_{ij}>b_i\}$.

Definition 2. The set $A=\{a_1,\ldots,a_n\}\subseteq V_m$ is called ***max-min linearly dependent*** if one of the vectors from A can be expressed as a max-min (min-max, respectively) linear combination of the others.

According to Definition 2 for testing max-min linear dependence of set of *n* vectors we have to solve *n* systems of the form (1).

3.TESTING LINEAR DEPENDENCE OVER FUZZY ALGEBRA

We develop four functions in MATLAB workspace for testing linear max-min dependence (independence, respectively) over the fuzzy algebra $([0,1], \max, \min)$. The first two functions follow conventional methods and algorithms [4, 12], while the proposed here functions are smaller time consuming.

In this paper our attention is paid on testing linear dependence in a set, see Definition 2. Conventional testing for max-min linear combination for members of $A = \{a_1, \ldots a_n\} \subseteq V_m$, according to Definition 2, requires to solve *n*-times a system of the form (1). Every time we have to omit a vector from the matrix A and to set it to the right-hand side of the system. If each system is inconsistent, then the set $A = \{a_1, \ldots a_n\} \subseteq V_m$ is max-min linearly independent, otherwise it is max-min linearly dependent.

3.1.Testing linear dependence by Sanchez formula [12]

When we test linear dependence by Sanchez formula, according to Th. 1, we compute $A^t \alpha B$. If the system $A \bullet X = B$ is consistent, it has unique maximum solution

$$X_{gr} = A^t \alpha B$$

and in this case the vector $B \in V_m$ is linear combination of the vectors from the set $A = \{a_1, \ldots a_n\} \subseteq V_m$. We run this function n times for each vector from $A = \{a_1, \ldots a_n\}$.

3.2.Algorithm A by Cechlárová and Plávka [2, 4]

The following Algorithm A is proposed in [2] for testing whether a fuzzy vector from $A = \{a_1, \ldots a_n\} \subseteq V_m$ is a max-min linear combination of the other vectors from $A = \{a_1, \ldots a_n\} \subseteq V_m$.

Algorithm A.

Step 1. Enter $A = \{a_1, \ldots a_n\} \subseteq V_m$.

Step 2. Compute A^t and $A^t \alpha A$.

Step 4. Initialize with zero's the main diagonal of $A^t \alpha A$. Write the result in C.

Step 5. Compute the $\max - \min$ composition $A \bullet C$.

Step 6. Compare each column of $A \bullet C$ with the corresponding column in A. If, say, the j-th column of $A \bullet C$ equals the j-th column of A, then the j-th vector of $A = \{a_1, \ldots a_n\} \subseteq V_m$ is a linear combination of the other vectors. The corresponding coefficients are the entries of the j-th column of C. If the equality does not occur for any $j = 1,\ldots,n$ then $A = \{a_1, \ldots a_n\} \subseteq V_m$ is linearly independent.

The code below reports whether $A = \{a_1, \ldots a_n\} \subseteq V_m$ is max-min linearly dependent or it is independent.

3.3.Modified Algorithm A by Peeva and Zahariev

Based on Theorem 3, we propose an improvement of Algorithm A. It concerns the former Steps 2 and 3, that will have now the form:

Step 2'. Compute C=($X_{gr}(j) = \min\limits_i \{b_i : a_{ij} > b_i\}$, j=1,…,n).

Step 3'. Initialize with zero's the main diagonal of C.

This improves the execution time for testing linear dependence, see fig. 1 and fig.2.

3.4.Testing max-min linear combination by Peeva [9]

The algorithm is based on solving fuzzy linear systems as presented in [9].

Algorithm

Step 1. Enter $A=\{a_1,\dots a_n\}\subseteq V_m$.

Step 2. Solve *n*-times a system of the form $A \bullet X = B$. Every time omit (initialize with zero's) a vector from the matrix A and set it to the right-hand side. If each system is inconsistent, then the set $A=\{a_1,\dots a_n\}\subseteq V_m$ is max-min linearly independent, otherwise it is max-min linearly dependent.

4. EXPERIMENTAL RESULTS

Following 4.1 – 4.4, we develop four functions in MATLAB workspace. The execution time of these algorithms is given on fig.1 and fig. 2. The lines are numbered as follows:

1. Program based on Sanchez formula, see 4.1.
2. Program based on Algorithm A, see 4.2.
3. Program based on Peeva's algorithm, see 4.4.
4. Program based on modified Algorithm A, see 4.3.

Fig. 1 concerns the case $A=\{a_1,\dots a_n\}\subseteq V_m$, when the number of vectors in the set A is n and each vector has m components.

- If the number m of the components of each vector is fixed (experiments are made when m=10, 20,…,500) and we increase the number n of the vectors from 5 to 500, the experimental results are shown in Fig. 1a. For fixed n, when the number of columns is large, the program based on 4.3 Modified algorithm A, works best.
- If the number n of the vectors is fixed (experiments are made when n=100, 200,…,10000) and we increase the number m of the components of each vector from 100 to 10000, the experimental results are shown in Fig. 1b. For fixed m, when the number of rows is large, the algorithm based on Peeva [9] works best;

Fig. 2 concerns the case when the number of vectors n in $A=\{a_1,\dots a_n\}$ equals the number of their components m, i.e. m=n, the matrix A_n is square. Then the program based on modified algorithm A works best.

5.CONCLUSIONS

This investigation is a step in finding the most efficient algorithm for testing linear dependency.

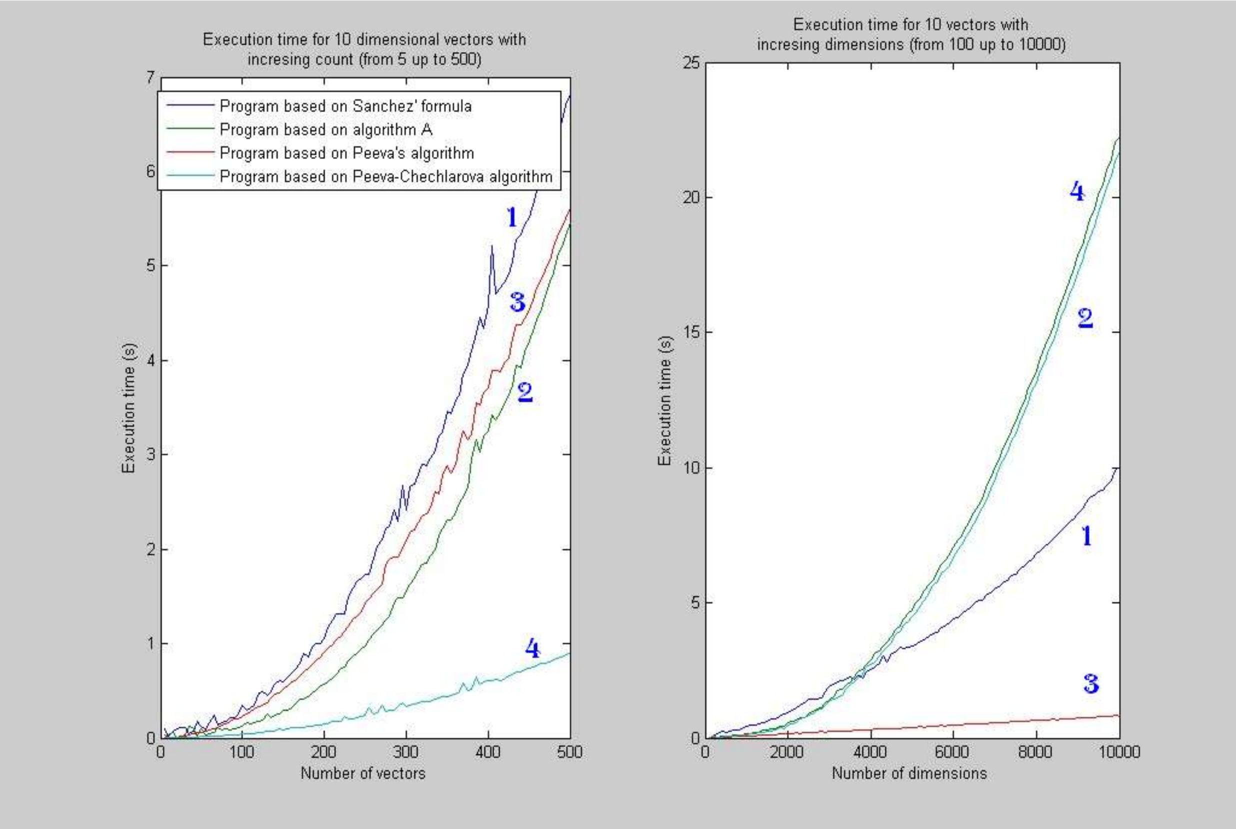

Fig. 1: Execution time for testing linear dependence in $A_{m \times n}$
a. m=10, n=5,…,500; b. m=100,…,10000, n=10

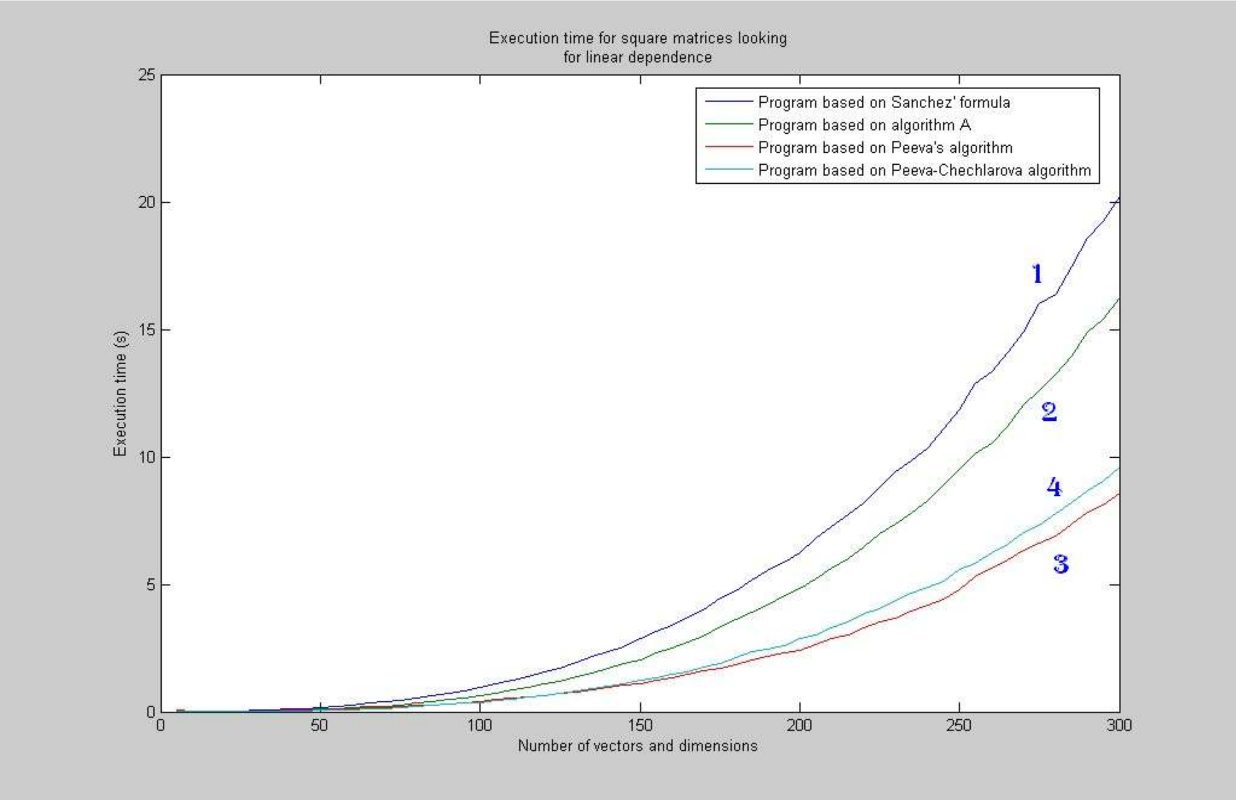

Fig. 2: Execution time for testing linear dependence in square A_n

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