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A Risk Model with Bivariate Pólya-Aeppli Counting Process

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Abstract. In this paper we introduce the bivariate Pólya-Aeppli process (BPAP) as a birth process. Then we consider the bivariate risk model with BPAP counting process. The ruin probability is discussed. In the case of exponentially distributed claims we derive partial differential equation for non-ruin probability and discuss the solution and properties.

INTRODUCTION

The Pólya-Aeppli process is a compound Poisson process with geometric compounding distribution. It was introduced in [1] and characterized in [2]. The construction of the Pólya-Aeppli process is quite simple and quite close to the classical case of counting processes. One relatively simple counting process is the sum of Poisson and Pólya-Aeppli process, see [3]. The simplicity of the Pólya-Aeppli process motivates the construction of the bivariate Pólya-Aeppli distribution. It was introduced in [4]. In this paper we introduce the bivariate Pólya-Aeppli process (BPAP). As application we consider the bivaraite risk model with BPAP counting process.

In the next section we introduce the process with extended bivariate Pólya-Aeppli distribution. The probability mass function is given with recursion formulas. The process is defined as a bivariate birth process and is applied as a counting process in bivariate risk model. The ruin probability with exponentially distributed claims is analyzed.

EXTENDED BIVARIATE PÓLYA-AEPPLI PROCESS

There are several approaches to extend an univariate distribution to bivariate form. In this section, we begin with the bivariate Poisson distribution obtained by the trivariate reduction method, see [5], and [6], and then compound this process with the geometric distribution, to derive a bivariate Pólya - Aeppli process.

Let $Z_i(t),\ i = 1,2,3,4$ be independent Pólya-Aeppli processes. Suppose that $Z_1(t) \sim PAP(\lambda_1,\rho_1), Z_2(t) \sim PAP(\lambda_2,\rho_2), Z_3(t) \sim PAP(\lambda_3,\rho_1), Z_4(t) \sim PAP(\lambda_3,\rho_2)$. Now, set

$$M(t) = Z_1(t) + Z_3(t) \quad \text{and} \quad N(t) = Z_2(t) + Z_4(t).$$

Then, according to the well known properties of the Pólya-Aeppli distribution, we have

$$M(t) \sim PAP(\lambda_1+\lambda_3,\rho_1) \quad \text{and} \quad N(t) \sim PAP(\lambda_2+\lambda_3,\rho_2).$$

This means that the compounding distribution of Z_1 and Z_3 is the geometric distribution with success probability $1-\rho_1$, and the compounding distribution of Z_2 and Z_4 is the geometric distribution with success probability $1-\rho_2$. Then the joint distribution of $(M(t),N(t))$ is the bivariate Pólya-Aeppli distribution, with the joint PGF as

$$\psi(t,s_1,s_2) = e^{-(\lambda_1+\lambda_2+\lambda_3)t} e^{\lambda_1 t\psi_1(s_1,\rho_1)+\lambda_2 t\psi_1(s_2,\rho_2)+\lambda_3 t\psi_1(s_1,\rho_1)\psi_1(s_2,\rho_2)}, \tag{1}$$

where $\psi_1(s,\rho)$ is the PGF of the geometric distribution, given by

$$\psi_1(s,\rho) = \frac{(1-\rho)s}{1-\rho s}. \tag{2}$$

Definition 1. *The probability distribution of $(M(t),N(t))$, corresponding to (1) and (2) is referred to as a bivariate Pólya - Aeppli distribution $(BivPA(\lambda_1,\lambda_2,\lambda_3,\rho_1,\rho_2))$, with parameters $\lambda_1,\lambda_2,\lambda_3,\rho_1$ and ρ_2.*

Remark 1. *If $\rho_1 = \rho_2$, the distribution in (1) reduces to the bivariate Pólya - Aeppli distribution, given in [4].*

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The marginal PGFs of $M(t)$ and $N(t)$ are easily obtained from (1), respectively, to be

$$\psi_{M(t)}(s_1) = \psi(t,s_1,1) = e^{-(\lambda_1+\lambda_3)t(1-\psi_1(s_1,\rho_1))}$$

and

$$\psi_{N(t)}(s_2) = \psi(t,1,s_2) = e^{-(\lambda_2+\lambda_3)t(1-\psi_1(s_2,\rho_2))}.$$

The means are given by $EM(t) = \frac{(\lambda_1+\lambda_3)t}{1-\rho_1}$ and $EN(t) = \frac{(\lambda_2+\lambda_3)t}{1-\rho_2}$, while the variances are $Var(M(t)) = \frac{(\lambda_1+\lambda_3)t(1+\rho_1)}{(1-\rho_1)^2}$ and $Var(N(t)) = \frac{(\lambda_2+\lambda_3)t(1+\rho_2)}{(1-\rho_2)^2}$. From (1), we obtain

$$\frac{\partial^2\Psi(t,s_1,s_2)}{\partial s_1 \partial s_2} = \Psi(t,s_1,s_2)[(\lambda_1 t + \lambda_3 t\psi_1(s_2,\rho_2))(\lambda_2 t + \lambda_3 t\psi_1(s_1,\rho_1)) + \lambda_3 t]\psi_1'(s_1,\rho_1)\psi_1'(s_2,\rho_2). \quad (3)$$

Upon substituting $s_1 = s_2 = 1$ in (3) and using the facts that $\psi_1(1) = 1$ and $\psi'(1) = EX = \frac{1}{1-\rho}$, we obtain the product moment of $M(t)$ and $N(t)$ to be

$$E(M(t)N(t)) = \frac{(\lambda_1+\lambda_3)t(\lambda_2+\lambda_3)t + \lambda_3 t}{(1-\rho_1)(1-\rho_2)},$$

which readily yields the covariance between N_1 and N_2 to be

$$Cov(M(t),N(t)) = \frac{\lambda_3 t}{(1-\rho_1)(1-\rho_2)} \quad (4)$$

and the correlation coefficient to be

$$Corr(M(t),N(t)) = \frac{\lambda_3}{\sqrt{(1+\rho_1)(1+\rho_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}}. \quad (5)$$

If $\rho_1 = \rho_2 = 0$, then $(M(t),N(t))$ has a bivariate Poisson distribution and the correlation coefficient is positive, (see [7] and [8] for example)

$$Corr(Y_1,Y_2) = \frac{\lambda_3}{\sqrt{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}}$$

and cannot exceed $\dfrac{\lambda_3}{\sqrt{\lambda_3+\min(\lambda_1,\lambda_2)}}$. Thus, from (5), we note that for the bivariate Pólya-Aeppli process $(M(t),N(t))$,

$$Corr(M(t),N(t)) < Corr(Y_1,Y_2).$$

Joint Probability Mass Function

Let $P(t,i,j) = P(M(t) = i, N(t) = j),\ i,j = 0,1,2,\ldots,$ be the joint probability mass function of $(M(t),N(t))$. The following proposition gives an extension of the Panjer recursion formulas, see [9].

Proposition 1. *The joint PMF of the bivariate Pólya-Aeppli process satisfies the following recursions:*

$$\begin{aligned} P(t,i,0) &= (2\rho_1 + \tfrac{(1-\rho_1)\lambda_1 t - 2\rho_1}{i})P(t,i-1,0) - (1-\tfrac{2}{i})\rho_1^2 P(t,i-2,0),\ i = 1,2,\ldots, \\ P(t,0,j) &= (2\rho_2 + \tfrac{(1-\rho_2)\lambda_2 t - 2\rho_2}{j})P(t,0,j-1) - (1-\tfrac{2}{j})\rho_2^2 P(t,0,j-2),\ j = 1,2,\ldots, \end{aligned} \quad (6)$$

and $P(t,-1,0) = 0, P(t,0,-1) = 0$. In addition,

$$P(t,i+1,j)-\rho_2 P(t,i+1,j-1)$$

$$=(2\rho_1+\tfrac{(1-\rho_1)\lambda_1 t-2\rho_1}{i+1})[P(t,i,j)-\rho_2 P(t,i,j-1)]-\rho_1^2(1-\tfrac{2}{i+1})[P(t,i-1,j)-\rho_2 P(t,i-1,j-1)]$$

$$+\frac{\lambda_3 t}{i+1}(1-\rho_1)(1-\rho_2)P(t,i,j-1)\ i=1,2,\ldots,j=1,2,\ldots,$$

$$P(t,i,j+1)-\rho_1 P(t,i-1,j+1)$$

$$=(2\rho_2+\tfrac{(1-\rho_2)\lambda_2 t-2\rho_2}{j+1})[P(t,i,j)-\rho_1 P(t,i-1,j)]-\rho_2^2(1-\tfrac{2}{j+1})[P(t,i,j-1)-\rho_1 P(t,i-1,j-1)]$$

$$+\frac{\lambda_3 t}{j+1}(1-\rho_1)(1-\rho_2)P(t,i-1,j),\ i=1,2,\ldots,j=1,2,\ldots$$

with $P(t,0,0)=e^{-(\lambda_1+\lambda_2+\lambda_3)t}$.

Proof.

Differentiation of (1) with respect to s_1 and s_2 yields

$$(1-\rho_1 s_1)^2(1-\rho_2 s_2)\frac{\partial\psi(t,s_1,s_2)}{\partial s_1}=(1-\rho_1)[\lambda_1 t+(\lambda_3 t-\rho_2(\lambda_1+\lambda_3)t)s_2]\psi(s_1,s_2) \tag{7}$$

and

$$(1-\rho_2 s_2)^2(1-\rho_1 s_1)\frac{\partial\psi(t,s_1,s_2)}{\partial s_2}=(1-\rho_2)[\lambda_2 t+(\lambda_3 t-\rho_1(\lambda_2+\lambda_3)t)s_1]\psi(s_1,s_2), \tag{8}$$

where $\psi(t,s_1,s_2)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P(M(t)=i,N(t)=j)s_1^i s_2^j$, $\frac{\partial\psi(t,s_1,s_2)}{\partial s_1}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)P(t,i+1,j)s_1^i s_2^j$ and $\frac{\partial\psi(t,s_1,s_2)}{\partial s_2}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)P(t,i,j+1)s_1^i s_2^j$. The required recursions are obtained by equating the coefficients of $s_1^i s_2^j$ on both sides, for fixed $i,j=0,1,2,\ldots$.

□

Theorem 1. *The joint probability mass function of the BPAP is given by*

$$P(t,i,0)=\sum_{m=1}^{i}\tbinom{i-1}{i-m}\tfrac{[\lambda_1 t(1-\rho_1)]^m}{j!}\rho_1^{i-m}P(t,0,0),\ \ i=1,2,\ldots,$$

$$P(t,0,j)=\sum_{l=1}^{j}\tbinom{j-1}{j-l}\tfrac{[\lambda_2 t(1-\rho_2)]^m}{m!}\rho_2^{j-l}P(t,0,0),\ \ j=1,2,\ldots,$$

$$\begin{aligned}P(t,i,j)&=\Bigg[\left(\sum_{m=1}^{i}\binom{i-1}{i-m}\frac{[\lambda_1 t(1-\rho_1)]^m}{m!}\rho_1^{i-m}\right)\left(\sum_{l=1}^{j}\binom{i-1}{l-1}\frac{[\lambda_2 t(1-\rho_2)]^l}{l!}\rho_2^{j-l}\right)\\&+\sum_{k=1}^{i\wedge j}\frac{[(1-\rho_1)(1-\rho_2)\lambda_3 t]^k}{k!}\sum_{m=0}^{i-k}\binom{i-1}{i-k-m}\frac{[\lambda_1 t(1-\rho_1)]^m}{m!}\rho_1^{i-k-m}\\&\times\sum_{l=0}^{j-k}\binom{j-1}{j-k-l}\frac{[\lambda_2 t(1-\rho_2)]^l}{l!}\rho_2^{j-k-l}\Bigg]P(t,0,0),\ \ i,j=1,2,\ldots,\end{aligned}$$

with $P(t,0,0)=e^{-(\lambda_1+\lambda_2+\lambda_3)t}$.

Proof. The initial value $P(t,0,0)=e^{-(\lambda_1+\lambda_2+\lambda_3)t}$ follows simply from (1), i.e., $P(t,0,0)=\psi(t,0,0)$. Formulas for $P(t,i,0),\ i=1,2,\ldots$ and $P(t,0,j),\ j=1,2,\ldots$ follow from (6). The PMF in the other cases are obtained recursively from the recursions stated in Proposition 1.

□

BIVARIATE BIRTH PROCESS

In this section we define the BPAP as a bivariate birth process. Suppose that for a small $h > 0$, we have the following assumptions.

$$\begin{aligned}
P(M(t+h)-M(t)=0, N(t+h)-N(t)=0) &= 1-(\lambda_1+\lambda_2+\lambda_3)h+o(h),\\
P(M(t+h)-M(t)=i, N(t+h)-N(t)=0) &= \lambda_1(1-\rho_1)\rho_1^{i-1}h+o(h), \ \ i=1,2,\ldots\\
P(M(t+h)-M(t)=0, N(t+h)-N(t)=j) &= \lambda_2(1-\rho_2)\rho_2^{j-1}h+o(h). \ \ j=1,2,\ldots\\
P(M(t+h)-M(t)=i, N(t+h)-N(t)=j) &= \lambda_3(1-\rho_1)(1-\rho_2)\rho_1^{i-1}\rho_2^{j-1}h+o(h), i,j=1,2,\ldots
\end{aligned}$$

The assumptions correspond to the definition of the bivariate Pólya-Aeppli distribution. From these assumptions, it follows that the probabilities $P(t,i,j),\ i,j=0,1,\ldots,\ t\geq 0$ satisfy the following equations

$$\frac{\partial}{\partial t}P(t,0,0)+(\lambda_1+\lambda_2+\lambda_3)P(t,0,0)=0,$$

$$\frac{\partial}{\partial t}P(t,i,0)+(\lambda_1+\lambda_2+\lambda_3)P(t,i,0)=\lambda_1(1-\rho_1)\sum_{m=0}^{i-1}\rho_1^{i-m-1}P(t,i,0), \ \ i=1,2,\ldots$$

$$\frac{\partial}{\partial t}P(t,0,j)+(\lambda_1+\lambda_2+\lambda_3)P(t,0,j)=\lambda_2(1-\rho_2)\sum_{n=0}^{j-1}\rho_2^{j-n-1}P(t,0,j), \ \ j=1,2,$$

For $i,j=1,2,\ldots,$ we have

$$\frac{\partial}{\partial t}P(t,i,j)+(\lambda_1+\lambda_2+\lambda_3)P(t,i,j)=\lambda_1(1-\rho_1)\sum_{m=0}^{i-1}\rho_1^{i-m-1}P(t,i,j-1)$$

$$+\lambda_2(1-\rho_2)\sum_{n=0}^{j-1}\rho_2^{j-n-1}P(t,i-1,j)+\lambda_3(1-\rho_1)(1-\rho_2)\sum_{m=0}^{i-1}\rho_1^{i-m-1}\sum_{n=0}^{j-1}\rho_2^{j-n-1}P(t,i,j).$$

APPLICATION TO RISK MODEL

Let us consider the following bivariate risk model.

$$U_1(t)=u_1+c_1t-\sum_{i=1}^{M(t)}X_i \quad \text{and} \quad U_2(t)=u_1+c_1t-\sum_{i=1}^{N(t)}Y_i.$$

Suppose that $X_i, i=1,2,\ldots$ are independent copies of $X\sim F_1(x)$, and $Y_i, i=1,2,\ldots$ are independent copies of $Y\sim F_2(x)$, with $F_1(0)=F_2(0)=0$. Additionally, (X,Y) is independent of $(M(t),N(t))$. Denote the means by $E(X)=\mu_1$, and $E(Y)=\mu_2$. The relative safety loadings are given by $\theta_i=\frac{c_i(1-\rho_i)}{\lambda_i\mu_i+\lambda_3(1-\rho_i)\mu_i}-1>0,\ \ i=1,2.$

Denote by $\tau_i=\inf\{t: U_i(t)<0\},\ \ i=1,2,$ the time to ruin of $U_i(t)$. The corresponding ruin probabilities are given by

$$\Psi_i(u_i)=P(\tau_i<\infty\,|\,U_i(t)=u_i),\ \ i=1,2.$$

We consider the following time to ruin

$$\tau_{min}=\min\{\tau_1,\tau_2\}=\inf\{t: U_1(t)<0 \text{ or } U_2(t)<0\},$$

and use the corresponding definition of ruin probability.

$$\begin{aligned}
\Psi(u_1,u_2) &= P(\tau_{min}<\infty\,|\,U_1(t)=u_1, U_2(t)=u_2)\\
&= P(\inf_{t\geq 0}\min(U_1(t),U_2(t))<0).
\end{aligned} \tag{9}$$

The ruin probability in (9) is analyzed in [10] and [11] in the case of bivariate Poisson counting process.

Denote by $\Phi(u_1,u_2)=1-\Psi(u_1,u_2)$ the non-ruin probability. For sufficiently small h, according to the assumptions in the previous section, we have the following.

$$\Phi(u_1,u_2)=[1-(\lambda_1+\lambda_2+\lambda_3)h]\Phi(u_1+c_1h,u_2+c_2h)$$

$$+\lambda_1(1-\rho_1)h\sum_{i=1}^{\infty}\rho_1^{i-1}\int_0^{u_1+c_1h}\Phi(u_1+c_1h-z_1,u_2)dF_1^{*i}(z_1)$$

$$+\lambda_2(1-\rho_2)h\sum_{j=1}^{\infty}\rho_1^{j-1}\int_0^{u_2+c_2h}\Phi(u_1,u_2+c_2h-z_2)dF_2^{*j}(z_2)$$

$$+\lambda_3(1-\rho_1)(1-\rho_2)h\sum_{i=1}^{\infty}\rho_1^{i-1}\sum_{j=1}^{\infty}\rho_2^{j-1}\int_0^{u_1+c_1h}\int_0^{u_2+c_2h}\Phi(u_1+c_1h-z_1,u_2+c_2h-z_2)dF_1^{*i}(z_1)dF_2^{*j}(z_2).$$

Denote by

$$H_1(z_1)=(1-\rho_1)\sum_{i=1}^{\infty}\rho_1^{i-1}F_1^{*i}(z_1)\ \text{ and }\ H_2(z_2)=(1-\rho_2)\sum_{j=1}^{\infty}\rho_2^{j-1}F_2^{*i}(z_2) \tag{10}$$

the distribution functions of the aggregated claims. The non-ruin probability $\Phi(u_1,u_2)$ satisfies the equation

$$\begin{aligned} c_1\tfrac{\partial}{\partial u_1}\Phi(u_1,u_2) &+c_2\tfrac{\partial}{\partial u_2}\Phi(u_1,u_2)=(\lambda_1+\lambda_2+\lambda_3)\Phi(u_1,u_2)\\ &-\lambda_1\textstyle\int_0^{u_1}\Phi(u_1-z_1,u_2)dH_1(z_1)-\lambda_2\int_0^{u_2}\Phi(u_1,u_2-z_2)dH_2(z_2)\\ &-\lambda_3\textstyle\int_0^{u_1}\int_0^{u_2}\Phi(u_1-z_1,u_2-z_2)dH_1(z_1)dH_2(z_2).\end{aligned} \tag{11}$$

EXPONENTIALLY DISTRIBUTED CLAIMS

In this case the distribution functions in (10) are again exponential, i.e.,

$$H_1(z_1)=1-e^{-\frac{1-\rho_1}{\mu_1}z_1}\ \text{ and }\ H_2(z_2)=1-e^{-\frac{1-\rho_2}{\mu_2}z_2}.$$

The equation (11) has the form:

$$\begin{aligned} c_1\tfrac{\partial^3}{\partial u_1^2\partial u_2}\Phi(u_1,u_2)+c_2\tfrac{\partial^3}{\partial u_1\partial u_2^2}\Phi(u_1,u_2) &=\left(\lambda_1+\lambda_2+\lambda_3-\tfrac{c_1(1-\rho_1)}{\mu_1}-\tfrac{c_2(1-\rho_2)}{\mu_2}\right)\tfrac{\partial^2}{\partial u_1\partial u_2}\Phi(u_1,u_2)\\ &-\tfrac{c_1(1-\rho_2)}{\mu_2}\tfrac{\partial^2}{\partial u_1^2}\Phi(u_1,u_2)-\tfrac{c_2(1-\rho_1)}{\mu_1}\tfrac{\partial^2}{\partial u_2^2}\Phi(u_1,u_2)\\ &+\tfrac{1-\rho_2}{\mu_2}\left(\lambda_1+\lambda_3-\tfrac{c_1(1-\rho_1)}{\mu_1}\right)\tfrac{\partial}{\partial u_1}\Phi(u_1,u_2)\\ &+\tfrac{1-\rho_1}{\mu_1}\left(\lambda_2+\lambda_3-\tfrac{c_2(1-\rho_2)}{\mu_2}\right)\tfrac{\partial}{\partial u_2}\Phi(u_1,u_2).\end{aligned} \tag{12}$$

One solution of this equation is given by:

$$\Phi(u_1,u_2)=1-\frac{1}{1+\theta_1}e^{-\frac{1-\rho_1}{\mu_1}\frac{\theta_1}{1+\theta_1}u_1}-\frac{1}{1+\theta_2}e^{-\frac{1-\rho_2}{\mu_2}\frac{\theta_2}{1+\theta_2}u_2}$$

$\Phi(u_1,u_2)$ satisfies the conditions $\Phi(\infty,\infty)=1$ and

$$\Phi(u_1,\infty)=\Phi_1(u_1)=1-\frac{1}{1+\theta_1}e^{-\frac{1-\rho_1}{\mu_1}\frac{\theta_1}{1+\theta_1}u_1}$$

and

$$\Phi(\infty,u_2)=\Phi_2(u_2)=1-\frac{1}{1+\theta_2}e^{-\frac{1-\rho_2}{\mu_2}\frac{\theta_2}{1+\theta_2}u_2},$$

where $\Phi_i(u_i), i = 1,2$ are the corresponding non-ruin probabilities of $U_i(t)$.

In the case of $\lambda_3 = 0$, i.e., when $M(t)$ and $N(t)$ are independent, the solution is given by

$$\Phi(u_1,u_2) = 1 - \frac{1}{1+\theta_1} e^{-\frac{1-\rho_1}{\mu_1}\frac{\theta_1}{1+\theta_1}u_1} - \frac{1}{1+\theta_2} e^{-\frac{1-\rho_2}{\mu_2}\frac{\theta_2}{1+\theta_2}u_2} + \frac{1}{1+\theta_1}\frac{1}{1+\theta_2} e^{-\frac{1-\rho_1}{\mu_1}\frac{\theta_1}{1+\theta_1}u_1} e^{-\frac{1-\rho_2}{\mu_2}\frac{\theta_2}{1+\theta_2}u_2}.$$

CONCLUSION

In this paper a risk model with a bivariate Pólya-Aeppli counting process is studied. An extended Bivariate Pólya-Aeppli process, joint probability mass function and Bivariate Pólya-Aeppli process as birh process are introduced. Essential part of this reserch is the application of the Bivariate Pólya-Aeppli process as a counting process in a bivariate risk model. A case of exponentially distributed claims is also given.

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