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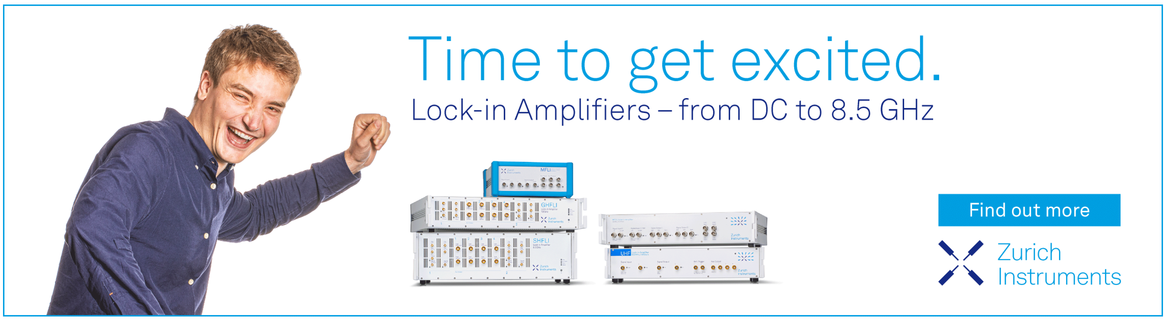

Fuzzy Relational Inequalities - Min-Goguen Implication

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Abstract. An algorithms for solving Min-Goguen fuzzy linear systems of inequalities is presented in this paper. Based on the logic for solving fuzzy linear systems of equations, we propose two separate algorithms for solving corresponding types of fuzzy linear systems of inequalities. Examples as well as analysis of the computational and memory complexity are also given.

INTRODUCTION

Systems of fuzzy relation equations and fuzzy relation inequalities, based on different fuzzy relational compositions are studied in details see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], in optimization problems with objective function subject to the fuzzy relation inequality constraints [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], in fuzzy logic systems with applications [26], in fuzzy mathematical programming [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40] and many other applied subjects.

This paper proposes methodology, unified method and algorithm for inverse problem resolution for Min-Goguen implication fuzzy linear systems of inequalities $A \rightarrow_{\odot} X \leq B$ (4) and $A \rightarrow_{\odot} X \geq B$ (5). The uniqueness of minimal solution of these systems is discussed. Then, the relationship between the system of inequalities and the corresponding fuzzy relation equation system $A \rightarrow_{\odot} X = B$ (6) is investigated. An algorithm for the fuzzy relation inequalities is presented. Simplification operations are given to accelerate the resolution of the problem by removing the components having no effect on the solution process. Also, an algorithm and some numerical examples are presented to illustrate the steps of the resolution. The main results are the algorithms for computing the complete solution set to the systems of fuzzy relation inequalities.

The paper is divided in seven sections. Next section gives some basic notions. Then, we briefly introduce algorithms for solving the associated system of fuzzy linear equations (6). The algorithms for solving the fuzzy linear systems of inequalities (4) and (5) are presented. Examples are also given. Analysis of the computational complexity and memory complexity of the algorithm as well as some conclusions can be found in the last two sections.

The terminology for fuzzy sets is according to [41], for fuzzy equations and inequalities – as in [1], [3], for algorithms, computational complexity and memory complexity – as in [42] and [43].

BASIC NOTIONS

We study:

$$\left| \begin{array}{l} (a_{11} \rightarrow_{\odot} x_1) \wedge (a_{12} \rightarrow_{\odot} x_2) \wedge \ldots \wedge (a_{1n} \rightarrow_{\odot} x_n) \leq b_1 \\ (a_{21} \rightarrow_{\odot} x_1) \wedge (a_{22} \rightarrow_{\odot} x_2) \wedge \ldots \wedge (a_{2n} \rightarrow_{\odot} x_n) \leq b_2 \\ \ldots \\ (a_{m1} \rightarrow_{\odot} x_1) \wedge (a_{m2} \rightarrow_{\odot} x_2) \wedge \ldots \wedge (a_{mn} \rightarrow_{\odot} x_n) \leq b_m \end{array} \right. \tag{1}$$

and

$$\left| \begin{array}{l} (a_{11} \rightarrow_{\odot} x_1) \wedge (a_{12} \rightarrow_{\odot} x_2) \wedge \ldots \wedge (a_{1n} \rightarrow_{\odot} x_n) \geq b_1 \\ (a_{21} \rightarrow_{\odot} x_1) \wedge (a_{22} \rightarrow_{\odot} x_2) \wedge \ldots \wedge (a_{2n} \rightarrow_{\odot} x_n) \geq b_2 \\ \ldots \\ (a_{m1} \rightarrow_{\odot} x_1) \wedge (a_{m2} \rightarrow_{\odot} x_2) \wedge \ldots \wedge (a_{mn} \rightarrow_{\odot} x_n) \geq b_m \end{array} \right. \tag{2}$$

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Also, for backing the presentation we will introduce the following fuzzy linear systems of equations:

$$\left| \begin{array}{l} (a_{11} \to_{\odot} x_1) \wedge (a_{12} \to_{\odot} x_2) \wedge \ldots \wedge (a_{1n} \to_{\odot} x_n) = b_1 \\ (a_{21} \to_{\odot} x_1) \wedge (a_{22} \to_{\odot} x_2) \wedge \ldots \wedge (a_{2n} \to_{\odot} x_n) = b_2 \\ \ldots \\ (a_{m1} \to_{\odot} x_1) \wedge (a_{m2} \to_{\odot} x_2) \wedge \ldots \wedge (a_{mn} \to_{\odot} x_n) = b_m \end{array} \right. \tag{3}$$

where $a_{ij}, b_i \in [0,1]$, are given and $x_j \in [0,1]$ marks the unknowns in the system. In this paper for the indices we suppose $i = 1,...,m,\ j = 1,...,n$

The system (1) will be presented in matrix form:

$$A \to_{\odot} X \leq B \tag{4}$$

The system (2) will be presented in matrix form:

$$A \to_{\odot} X \geq B \tag{5}$$

The system (3) will be presented in matrix form:

$$A \to_{\odot} X = B \tag{6}$$

where $A = (a_{ij})_{m\times n}$ is the matrix of coefficients, $B = (b_i)_{m\times 1}$ holds for the right-hand side vector and $X = (x_j)_{1\times n}$ is the vector of unknowns.

Let $a, b \in [0,1]$.

Operation $\vee$ between a and b is defined as

$$a \vee b = max(a,b) \tag{7}$$

Operation $\wedge$ between a and b is defined as

$$a \wedge b = min(a,b) \tag{8}$$

Operation $\odot$ between a and b is defined as the conventional real numbers multiplication.

Operation $\to_{\odot}$ between a and b is defined as

$$a \to_{\odot} b = \begin{cases} 1, \text{ if } a \leq b \\ \frac{b}{a}, \text{ if } a > b \end{cases} \tag{9}$$

A matrix $A = (a_{ij})_{m\times n}$ with $a_{ij} \in [0,1]$ for each $i = 1,...,m,\ j = 1,...,n$ is called *membership matrix*. In what follows *'matrix'* is used instead of *'membership matrix'*.

Let the matrices $A = (a_{ij})_{m\times p}$ and $B = (b_{ij})_{p\times n}$ be given.

The matrix $C_{m\times n} = (c_{ij}) = A \to_{\odot} B$ is called *min*$- \to_{\odot}$ *product* of A and B if

$$c_{ij} = \wedge_{k=1}^{p} (a_{ik} \to_{\odot} b_{kj}) \tag{10}$$

for each $i = 1,...,m,\ j = 1,...,n$.

The matrix $C_{m\times n} = (c_{ij}) = A \odot B$ is called *max* $- \odot$ *product* of A and B if

$$c_{ij} = \vee_{k=1}^{p} (a_{ik} \odot b_{kj}) \tag{11}$$

for each $i = 1,...,m,\ j = 1,...,n$.

For $X = (x_j)_{1\times n}$ and $Y = (y_j)_{1\times n}$ the inequality $X \geq Y$ holds iff $x_j \geq y_j$ for each $j = 1,...,n$.

Next notions are according to [4]

A vector $X^0 = (x_j^0)_{1\times n}$ with $x_j^0 \in [0,1],\ j = 1,...,n$, is called *solution* of the systems (4), (5), and (6) if $A \to_{\odot} X^0 = B$ holds. The set of all solutions of a system is called *complete solution set* and it is denoted by $\mathbb{X}^0$. If $\mathbb{X}^0 \neq \emptyset$ then the system is called *solvable* (or *consistent*), otherwise it is called *unsolvable* (or *inconsistent*).

A solution $X_u^0 \in \mathbb{X}^0$ is called *upper solution* if for any $X^0 \in \mathbb{X}^0$ the inequality $X_u^0 \leq X^0$ implies $X^0 = X_u^0$. A solution $X_{low}^0 \in \mathbb{X}^0$ is called *lower solution* if for any $X^0 \in \mathbb{X}^0$ the inequality $X^0 \leq X_{low}^0$ implies $X^0 = X_{low}^0$. If the lower solution is unique, it is called *lowest* (or *minimum*) solution. The n-tuple $(X_1,...,X_n)$ with $X_j \subseteq [0,1]$ is called *interval solution* if any $X^0 = (x_j^0)_{n\times 1}$ with $x_j^0 \in X_j$ for each $j = 1,...,n$ implies $X^0 = (x_j^0)_{n\times 1} \in \mathbb{X}^0$. Any interval solution whose components (interval bounds) are determined by the lowest solution from the left and by an upper solution from the right, is called *minimal interval solution* of (4), (5), and (6).

SOLVING $A \rightarrow_{\odot} X = B$

Methods to solve similar to (6) systems are investigated [1] and [8] where it can be spotted that there is some level of similarity between a whole class of such a fuzzy linear system of equations. While this similarity is still to be generalizes in a more universal algorithm, it is still possible to construct an algorithm for (6) based on the general logical construction of the algorithms in [1] and [8]. As here we are more interested on solving (4) and (5), will not focus on it here. Example will be partially presented further, in it's parts which contributes for the fuzzy linear system of inequalities solution.

It is important that in order to solve fuzzy linear systems of inequalities (4) and (5) we also need methods and algorithms to solve fuzzy linear systems of equations (6). In this article we proof that there is a direct connection between the solutions of (6) and the solutions of (4) and (5). Depending on the type of the inequalities ($\leq$ or $\geq$) we take one or another part of the algorithms need to solve (6) and then add on top of it to find the full set of interval solutions for both (4) and 5

SOLVING $A \rightarrow_{\odot} X \leq B$ AND $A \rightarrow_{\odot} X \geq B$

It is well known [1], [8], that any solvable $min- \rightarrow_{\odot}$ fuzzy linear system of equations has unique lowest solution and can have many upper solutions. In order to find all solutions of the solvable system (6), it is necessary to find both its lowest solution and all of its upper solutions.

In order to solve (4) and (5) we also need to take into consideration that:

Theorem 1. A solvable system $A \rightarrow_{\odot} X \leq B$ has its lowest solution $\check{X} = (\check{x}_j)$ where $\check{x}_j = 0$ for $j = 1, \ldots, n$.

Proof. From (9) we can see that for every component $a_{ij} \rightarrow_{\odot} x_j$ from the system, if we have $x_j = 0$, the whole component will become 0. From (1) for every inequality from the system, we take minimum between all those components, which makes the left side of the inequality $= 0$, which is $\leq b_j$, because by definition b_j cannot be lower than 0. □

Theorem 2. A solvable system $A \rightarrow_{\odot} X \geq B$ has unique upper solution $\hat{X} = (\hat{x}_j)$ where $\hat{x}_j = 1$ for $j = 1, \ldots, n$.

Proof. From (9) we can see that for every component $a_{ij} \rightarrow_{\odot} x_j$ from the system, if we have $x_j = 1$, the whole component will become 1. From (1) for every inequality from the system, we take minimum between all those components, which makes the left side of the inequality $= 1$, which is $\geq b_j$, because by definition b_j cannot be higher than 1. □

Based on [1], Theorem 1, and Theorem 2, we can now define the following algorithms for solving (4) and (5).

Algorithm 1 Solve (4).

1. Obtain input data for the matrices A and B.
2. Set the lowest solution for the system $\check{X} = (\check{x}_j)$ where $\check{x}_j = 0$ for $j = 1, \ldots, n$, and check it for consistency.
3. If the system is unsolvable go to step 6.
4. Obtain all maximal solutions for the system (6)
5. Exit.

The number of interval solutions for the system (4) is the same as the number its maximal solutions. They are all defined by $\check{X}$ on the left and one of the upper solutions on the right.

Algorithm 2 Solve (5).

1. Obtain input data for the matrices A and B.
2. Obtain the lowest solution for the system (6)
3. If the system is unsolvable go to step 6.
4. Set a unique maximal solution for the system $\hat{X} = (\hat{x}_j)$ where $\hat{x}_j = 1$ for $j = 1, \ldots, n$.
5. Exit.

The system (5) has a single interval solution. It is defined by $\check{X}$ on the left and $\hat{X}$ on the right.

EXAMPLES

In the first example we illustrate presented algorithms. The next examples shows solving problems with a software based on the presented here algorithms and developed by Zl. Zahariev. Execution time is also given.

Example 1

Using (Algorithm 2), solve

$$\begin{pmatrix} 0 & 0.1 & 0.8 & 0.3 & 0 & 0.5 \\ 0.2 & 0.6 & 0.48 & 0 & 0 & 0.3 \\ 0.05 & 0.3 & 0.24 & 0 & 0.12 & 0 \\ 0 & 0 & 0.48 & 0.4 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0 & 0.8 & 0.48 & 0.6 \\ 0.1 & 0.3 & 0.24 & 0.2 & 0.12 & 0.15 \end{pmatrix} \rightarrow_{\odot} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 0.3 \\ 0.5 \\ 1 \\ 0.5 \\ 0.25 \\ 1 \end{pmatrix} \tag{12}$$

Finding the lowest solution

The lowest solution for (12) is the same as the lowest solution for the system:

$$\begin{pmatrix} 0 & 0.1 & 0.8 & 0.3 & 0 & 0.5 \\ 0.2 & 0.6 & 0.48 & 0 & 0 & 0.3 \\ 0.05 & 0.3 & 0.24 & 0 & 0.12 & 0 \\ 0 & 0 & 0.48 & 0.4 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0 & 0.8 & 0.48 & 0.6 \\ 0.1 & 0.3 & 0.24 & 0.2 & 0.12 & 0.15 \end{pmatrix} \rightarrow_{\odot} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.5 \\ 1 \\ 0.5 \\ 0.25 \\ 1 \end{pmatrix} \tag{13}$$

Using an algorithm for finding the lowest solution of (6), we can obtain the lowest solution of (13). It is:

$$\check{X} = (0.1\ \ 0.3\ \ 0.24\ \ 0.2\ \ 0.12\ \ 0.15)'$$

Upper solutions

From Theorem 2 we set a single upper solution $X_{u_1} = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)'$

Example 2

Solve the same system:

$$\begin{pmatrix} 0 & 0.1 & 0.8 & 0.3 & 0 & 0.5 \\ 0.2 & 0.6 & 0.48 & 0 & 0 & 0.3 \\ 0.05 & 0.3 & 0.24 & 0 & 0.12 & 0 \\ 0 & 0 & 0.48 & 0.4 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0 & 0.8 & 0.48 & 0.6 \\ 0.1 & 0.3 & 0.24 & 0.2 & 0.12 & 0.15 \end{pmatrix} \rightarrow_{\odot} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \leq \begin{pmatrix} 0.3 \\ 0.5 \\ 1 \\ 0.5 \\ 0.25 \\ 1 \end{pmatrix}$$

using developed by Zl. Zahariev software. This also will demonstrate the efficiency of the presented here algorithm. The software used in this example as well as instructions can be found in [44].

Solving the problem

1. Input matrix A and B
2. Declare A and B as fuzzy matrices. *FuzzyMatrix* is a MATLAB class holding some crucial operations for a fuzzy matrices ([22])

```
>> A=fuzzyMatrix(A); B=fuzzyMatrix(B);
```

3. Create the system object with 'goguen' composition, empty matrix X, an option for finding all upper solutions set to 'true', and an option to solve the $\leq$ type of system set as a $inequalities = -1$

```
>> S = fuzzySystem('goguen', A, B, fuzzyMatrix(), true, -1)
```

4. Solve the system

```
>> S.solve_inverse()
ans =
  fuzzySystem with properties:
     composition: 'goguen'
               a: [6?6 fuzzyMatrix]
               b: [6?1 fuzzyMatrix]
               x: [1?1 struct]
            full: 1
    inequalities: -1
```

5. The member variable *x* is a stricture which holds all the solutions of the system. We can inspect it and see that among the other information it holds one lower solution (*x.low*) and 5 upper solution (*x.gr*)

```
>> S.x
ans =
  struct with fields:
         rows: 6
         cols: 6
         help: [3?6 fuzzyMatrix]
          low: [6?1 fuzzyMatrix]
          ind: [6?1 double]
        exist: 1
    dominated: [2 3 6]
    help_rows: 3
           gr: [6?5 fuzzyMatrix]

>> S.x.low
ans =
  6?1 fuzzyMatrix:
  double data:
     0
     0
     0
     0
     0
     0

>> S.x.gr
ans =
```

```
6?5 fuzzyMatrix:
double data:
  0.1000    1.0000    1.0000    1.0000    1.0000
  1.0000    1.0000    1.0000    1.0000    1.0000
  0.2400    0.2400    0.2400    0.2400    1.0000
  1.0000    0.2000    1.0000    1.0000    0.2000
  1.0000    1.0000    0.1200    1.0000    1.0000
  1.0000    1.0000    1.0000    0.1500    0.1500
```

Example 3

Solve the same system:

$$\begin{pmatrix} 0 & 0.1 & 0.8 & 0.3 & 0 & 0.5 \\ 0.2 & 0.6 & 0.48 & 0 & 0 & 0.3 \\ 0.05 & 0.3 & 0.24 & 0 & 0.12 & 0 \\ 0 & 0 & 0.48 & 0.4 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0 & 0.8 & 0.48 & 0.6 \\ 0.1 & 0.3 & 0.24 & 0.2 & 0.12 & 0.15 \end{pmatrix} \rightarrow_{\odot} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 0.3 \\ 0.5 \\ 1 \\ 0.5 \\ 0.25 \\ 1 \end{pmatrix}$$

Solving the problem

1. Input matrix A and B
2. Declare A and B as fuzzy matrices. *FuzzyMatrix* is a MATLAB class holding some crucial operations for a fuzzy matrices ([22])

   ```
   >> A=fuzzyMatrix(A); B=fuzzyMatrix(B);
   ```

3. Create the system object with 'goguen' composition, empty matrix X, an option for finding all upper solutions set to 'true', and an option to solve the $\geq$ type of system set as a $inequalities = 1$

   ```
   >> S = fuzzySystem('goguen', A, B, fuzzyMatrix(), true, 1)
   ```

4. Solve the system

   ```
   >> S.solve_inverse()
   ans =
     fuzzySystem with properties:
        composition: 'goguen'
                  a: [6?6 fuzzyMatrix]
                  b: [6?1 fuzzyMatrix]
                  x: [1?1 struct]
               full: 1
       inequalities: 1
   ```

5. The member variable x is a stricture which holds all the solutions of the system. We can inspect it and see that among the other information it holds one lower solution ($x.low$) and one upper solution ($x.gr$)

   ```
   >> S.x
   ans =
     struct with fields:
   ```

```
     rows: 6
     cols: 6
     help: [6?6 fuzzyMatrix]
      low: [6?1 fuzzyMatrix]
      ind: [6?1 double]
    exist: 1
       gr: [6?1 fuzzyMatrix]

>> S.x.low
ans =
  6?1 fuzzyMatrix:
  double data:
    0.1000
    0.3000
    0.2400
    0.2000
    0.1200
    0.1500

>> S.x.gr
ans =
  6?1 fuzzyMatrix:
  double data:
     1
     1
     1
     1
     1
     1
```

Execution time

On a test computer Example 2 was solved with the presented above software in about 0.001 seconds, Example 3 took only 0.0007 seconds.

COMPUTATIONAL AND MEMORY COMPLEXITY

In this section the terminology for computational complexity and for memory complexity is according to [42].

Solving (4) (Algorithm 1) has exponential time complexity. However, it's order depends not on the size of the system, but on the number of the solutions.

Solving (5) (Algorithm 2) has time complexity $O(m.n)$ and memory complexity for this part of the $O(m.n+m+n)$.

CONCLUSION

Presented here algorithms and software are fast and reliable way to find all solution of Min-Goguen fuzzy linear systems of inequalities. More information about the developed software can be found at [44], [45].

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