Existence and Uniqueness of a Periodic Solution of 3-Conductor Transmission Line System with Nonlinear Resistive Loads

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***Abstract*—This paper is part two of the study of the electromagnetic compatibility characteristics of lossless transmission lines terminated by resistive polynomial type of nonlinear loads initially proposed by C. Paul. Based on the obtained system of two functional equations and two neutral equations for four unknown functions in part 1 of our analysis, we proved that the mixed problem is equivalent to an initial value problem for a functional system on the boundary. The system of functional equations is solved by fixed point method. Commonly, these problems are solved by numerical methods or by Laplace transformation method that is valid only for linear problems. Our results are verified versus real-world example. The method here proposed could be applied for nonlinear boundary conditions too.**

Keywords—lossless transmission lines, nonlinear resistive loads, mixed problem

I. INTRODUCTION

Our investigation of electromagnetic compatibility (EMC) of printed circuit boards (PCB) based on lossless transmission lines terminated by nonlinear loads is split in two parts. The 3-conductor transmission lines setup is taken from C. Paul results in [1]. Distinct to [1] we propose a broader treatment to find a solution to the system describing the 3-conductor transmission line problem.

In part 1 [2] of our study, we formulated a hyperbolic system modeling the behavior of 3-conductor transmission line terminated by nonlinear resistive loads. Then we transformed the system to a diagonal form using the method from [3]; we also transformed the initial and boundary conditions. The reduced system is consisting of two functional equations and two equations of neutral type for four unknown functions – these are the currents and voltages of the transmission line. The mixed problem for the diagonal system is reduced to initial problem on the boundary. The system of functional equations is solved by fixed point method. Commonly, such problems are solved by numerical methods or by Laplace transformation method that is valid only for linear problems.

In this paper, which is part 2 of our study, proceeding form the obtained system of two functional equations and two neutral equations obtained in part 1 of the analysis, we prove that the mixed problem is equivalent to an initial value problem for a functional system on the boundary.

For the sake of completeness, we note that in [4] we investigated the case of the lines being terminated by linear loads, while in [2] as well as in this paper, we consider the case of the lines being terminated by nonlinear resistive loads.

II. OPERATOR PRESENTATION OF THE PERIODIC PROBLEM

In part 1 of our analysis [2] we reduced the mixed problem for hyperbolic system to initial value problem for a system consisting of two neutral equations and two functional equations on the boundary.

$$I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)$$

Introduce the sets

$$M_1 = \left\{ I_1(t) \in C_{T_0}[0,\infty) : \left|I_1(t)\right| \le I_{01} e^{\mu(t-kT_0)},\ \ t \in [kT_0,(k+1)T_0] \right\}$$

$$M_2 = \left\{ I_2(t) \in C_{T_0}[0,\infty) : \left|I_2(t)\right| \le I_{02} e^{\mu(t-kT_0)},\ t \in [kT_0,(k+1)T_0] \right\}$$

$$M_3 = \left\{ I_3(t) \in C_{T_0}[0,\infty) : \left|I_3(t)\right| \le I_{03} e^{\mu(t-kT_0)},\ \ t \in [kT_0,(k+1)T_0] \right\}$$

$$M_4 = \left\{ I_4(t) \in C_{T_0}[0,\infty) : \left|I_4(t)\right| \le I_{04} e^{\mu(t-kT_0)},\ t \in [kT_0,(k+1)T_0] \right\}$$

where $C_{T_0}[0,\infty)$ is the set of all continuous T_0-periodic functions and I_{0k}, T_0, μ are positive constants and $\mu T_0 = \mu_0 =$ const.

We use the technique of fixed-point theory in uniform spaces (cf. [3]). For that purpose, we introduce a saturated family of pseudo-metrics in the Cartesian product

$$M = M_1 \times M_2 \times M_3 \times M_4$$

$$\rho_k(I_n, \bar{I}_n) = \max\left\{ \left|I_n(t) - \bar{I}_n(t)\right| e^{-\mu(t-kT_0)} : t \in [kT_0,(k+1)T_0] \right\}$$

$$(n = 1,2,3,4;\ k = 0,1,2,\ldots)$$

where the index set of this family consists of all ordered fours

$$\left(p_1, p_2, p_3, p_4\right) \in N_0 \times N_0 \times N_0 \times N_0;\ N_0 = \{0,1,2,\ldots\}$$

corresponding to the initial points of the intervals

$$[p_1T_0,(p_1+1)T_0] \times [p_2T_0,(p_2+1)T_0] \times$$
$$\times [p_3T_0,(p_3+1)T_0] \times [p_4T_0,(p_4+1)T_0]$$

The set M turns out into a complete uniform space with a saturated family of pseudometrics

$$\rho_{(p_1,p_2,p_3,p_4)}\left((I_1,I_2,I_3,I_4),(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4)\right) =$$
$$= \max\left\{ \rho_{p_1}(I_1,\bar{I}_1), \rho_{p_2}(I_2,\bar{I}_2), \rho_{p_3}(I_3,\bar{I}_3), \rho_{p_4}(I_4,\bar{I}_4) \right\}$$

Introduce maps $j_n : N_0 \to N_0 \quad (n = 1,2)$ in the following way:

$$j_1 : N_0 \to N_0 ;\ [kT_0,(k+1)T_0] \to [kT_0 - T_1,(k+1)T_0 - T_1] ;$$

We suppose that $T_p = m_p T_0 (p = 1,2)$. Therefore

$$[kT_0 - T_p, (k+1)T_0 - T_p] = [(k - m_p)T_0, (k+1-m_p)T_0]$$

and then

$$j_p : k \to k - m_p$$

provided

$$k - m_p \ge 0;\ j_p{}^m(k) = j_p(j_p^{m-1}(k)),\ j_p{}^0(k) = k.$$

The definition of j_p implies that $j_p{}^m(k) \in N_0$ only for finite m. Then we define the map

$$j(p_1, p_2, p_3, p_4) : N_0 \times N_0 \times N_0 \times N_0 \to N_0 \times N_0 \times N_0 \times N_0$$

in the proof of the main theorem.

Now we ***formulate the main problem***: to find a T_0-periodic solution of the neutral system (cf. equations (19)-(22) in [2]) we define an operator with components $B = (B_1, B_2, B_3, B_4)$ by the formulas:

$$B_k{}^{(1)}(I_1, I_2, I_3, I_4)(t) := A_{10}(t) + A_{11}I_3(t - T_1) + A_{12}I_4(t - T_2),$$
$$t \in [T + kT_0, T + (k+1)T_0]$$

$$B_k{}^{(2)}(I_1, I_2, I_3, I_4)(t) := A_{20}(t) + A_{21}I_3(t - T_1) + A_{22}I_4(t - T_2),$$
$$t \in [T + kT_0, T + (k+1)T_0]$$

$$B_k{}^{(3)}(I_1, I_2, I_3, I_4)(t) := \int_{T+kT_0}^{t} U_3(I_1, I_2, I_3, I_4)(s)ds -$$
$$- \frac{t - T - kT_0}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} U_3(I_1, I_2, I_3, I_4)(s)ds,$$
$$t \in [T + kT_0, T + (k+1)T_0]$$

$$B_k{}^{(4)}(I_1, I_2, I_3, I_4)(t) := \int_{T+kT_0}^{t} U_4(I_1, I_2, I_3, I_4)(s)ds -$$
$$- \frac{t - T - kT_0}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} U_4(I_1, I_2, I_3, I_4)(s)ds,$$
$$t \in [T + kT_0, T + (k+1)T_0]$$

where the functions $I_1(t - T_1), I_2(t - T_1), I_3(t - T_1), I_4(t - T_2)$ are substituted by corresponding initial functions.

Lemma 1. The function

$$B_k{}^{(n)}(I_1, I_2, I_3, I_4)(t)\ (n = 1,2,3,4)$$

for $t \in [T + kT_0, T + (k+1)T_0]$ equals $B_{k+1}{}^{(n)}(I_1, I_2, I_3, I_4)(t)$ for $t \in [T + (k+1)T_0, T + (k+2)T_0]$, that is,

$$B_k{}^{(n)}(I_1, I_2, I_3, I_4)(t - T_0) = B_{k+1}{}^{(n)}(I_1, I_2, I_3, I_4)(t),$$
$$t \in [T + (k+1)T_0, T + (k+2)T_0];\ (n = 3, 4)$$

The proof is analogous to the one from [3].

Lemma 2. The periodic problem (cf. equations (19)-(22) in [2]) has a unique solution

$$(I_1(.), I_2(.), I_3(.), I_4(.)) \in M_1 \times M_2 \times M_3 \times M_4$$

iff the operator

$$B = \begin{pmatrix} B^{(1)}(I_1, I_2, I_3, I_4), B^{(2)}(I_1, I_2, I_3, I_4), \\ B^{(3)}(I_1, I_2, I_3, I_4), B^{(4)}(I_1, I_2, I_3, I_4) \end{pmatrix}$$

has a fixed point, that is,

$$(I_1, I_2, I_3, I_4) = \begin{pmatrix} B^{(1)}(I_1, I_2, I_3, I_4), B^{(2)}(I_1, I_2, I_3, I_4), \\ B^{(3)}(I_1, I_2, I_3, I_4), B^{(4)}(I_1, I_2, I_3, I_4) \end{pmatrix}$$

The proof of Lemma 2 is done following the approach in [3].

III. Existence-Uniqueness of Periodic Solution

The main result is:

Theorem 1. Let the following conditions be fulfilled:

$$\begin{aligned} & I_{10}(.), I_{30}(.) \in C^1_{T_0}[-T_1, 0], \\ & I_{20}(.), I_{40}(.) \in C^1_{T_0}[-T_2, 0] \\ & U_S(.) \in C_{T_0}[0, \infty), \\ & \breve{U}_S = \max\{|U_S(t)| : t \in [0, T_0]\} \end{aligned} \qquad (1)$$

Assumptions (D) and (L) are valid;
$T_1 = m_1T_0$, $T_2 = m_2T_0$ (2)
for some positive integers m_1, m_2 .

$$\begin{aligned} & \frac{|q_2 + R_{NE}\gamma_2|}{|\Delta_{12}|}\breve{U}_S = \\ & = \frac{1}{|\Delta_{12}|}\left| \begin{matrix} \lambda_2 L_{12} + (L_{22}\lambda_2 + R_{NE})\times \\ \times \dfrac{1 - \lambda_2{}^2(L_{11}C_{11} + L_{12}C_{12})}{\lambda_2{}^2(L_{12}C_{11} + L_{22}C_{12})} \end{matrix} \right| \breve{U}_S < \\ & < \min\{I_{10}, I_{20}\} \end{aligned} \qquad (3)$$

$$\begin{aligned} & \frac{|q_1 + R_{NE}\gamma_1|}{|\Delta_{12}|}\breve{U}_S = \\ & = \frac{1}{|\Delta_{12}|}\left| \begin{matrix} \lambda_1 L_{12} + (L_{22}\lambda_1 + R_{NE})\times \\ \times \dfrac{1 - \lambda_1{}^2(L_{11}C_{11} + L_{12}C_{12})}{\lambda_1{}^2(L_{12}C_{11} + L_{22}C_{12})} \end{matrix} \right| \breve{U}_S < \\ & < \min\{I_{10}, I_{20}\} \end{aligned} \qquad (4)$$

Then there exists a unique-periodic solution of eq. (4) in [2].

Proof: The set $M_1 \times M_2 \times M_3 \times M_4$ is a uniform space with the above saturated family of pseudo-metrics. In view of $\lambda_1 > \lambda_2 \Rightarrow \Lambda/\lambda_2 > \Lambda/\lambda_1 \Rightarrow T_2 > T_1$ we show that B maps $M_1 \times M_2 \times M_3 \times M_4$ into itself. It is easy to verify that all components of the operator B are periodic functions.

For $t \in [kT_0, (k+1)T_0]$ and for sufficiently large $\mu > 0$ we obtain

$$\left|B_k^{(1)}(I_1,I_2,I_3,I_4)(t)\right| \le$$
$$\le |A_{10}(t)| + |A_{11}||I_3(t-T_1)| + |A_{12}||I_4(t-T_2)| \le$$
$$\le |A_{10}(t)| + |A_{11}| I_{30} e^{\mu(t-T_1-kT_0)} + |A_{12}| I_{40} e^{\mu(t-T_2-kT_0)} \le$$
$$\le e^{\mu(t-kT_0)}\left(\frac{|q_2+R_{NE}\gamma_2|}{|\Delta_{12}|}\breve{U}_S + |A_{11}| I_{30} e^{-\mu T_1} + |A_{12}| I_{40} e^{-\mu T_2}\right) \le$$
$$\le e^{\mu(t-kT_0)} I_{10}$$

$$\left|B_k^{(2)}(I_1,I_2,I_3,I_4)(t)\right| \le$$
$$\le |A_{20}(t)| + |A_{21}||I_3(t-T_1)| + |A_{22}||I_4(t-T_2)| \le$$
$$\le \frac{|q_1+R_{NE}\gamma_1|\breve{U}_S}{|\Delta_{12}|} + |A_{21}| I_{30} e^{\mu(t-T_1-kT_0)} + |A_{22}| I_{40} e^{\mu(t-T_2-kT_0)} \le$$
$$\le e^{\mu(t-kT_0)}\left(\frac{|q_1+R_{NE}\gamma_1|\breve{U}_S}{|\Delta_{12}|} + |A_{21}| I_{30} e^{-\mu T_1} + |A_{22}| I_{40} e^{-\mu T_2}\right) \le$$
$$\le e^{\mu(t-kT_0)} I_{20}$$

$$\left|B_k^{(3)}(I_1,I_2,I_3,I_4)(t)\right| \le I_{10} e^{-\mu T_1} e^{\mu(t-kT_0)} +$$
$$+\frac{|p_2\gamma_1-q_2|}{\Delta_{34}C_0} I_{10} e^{-\mu T_1} e^{\mu(t-kT_0)} +$$
$$+\frac{|p_2\gamma_2-q_2|}{\Delta_{34}C_0} I_{20} e^{-\mu T_2}\, \frac{e^{\mu(t-kT_0)}-1}{\mu} +$$
$$+\frac{|p_2\gamma_1-q_2|}{\Delta_{34}C_0} I_{30}\, \frac{e^{\mu(t-kT_0)}-1}{\mu} +$$
$$+\frac{|p_2\gamma_2-q_2|}{\Delta_{34}C_0} I_{40}\, \frac{e^{\mu(t-kT_0)}-1}{\mu} +$$
$$+\frac{|q_2|}{\Delta_{34}C_0}\sum_{p=1}^{m}\left|g_p^1\right|\begin{pmatrix}|p_1| I_{10} e^{-\mu T_1} + \\ +|p_2| I_{20} e^{-\mu T_2} + \\ +|p_1| I_{30} + |p_2| I_{40}\end{pmatrix}^p \int_{T+kT_0}^{t} e^{p\mu(s-kT_0)} ds +$$
$$+\frac{|p_2|}{\Delta_{34}C_0}\sum_{p=1}^{m}\left|g_p^2\right|\begin{pmatrix}|q_1| I_{10} e^{-\mu T_1} + \\ +|q_2| I_{20} e^{-\mu T_2} + \\ +|q_1| I_{30} + |q_2| I_{40}\end{pmatrix}^p \int_{T+kT_0}^{t} e^{p\mu(s-kT_0)} ds \le$$
$$\le \frac{e^{\mu(t-kT_0)}}{\Delta_{34}C_0}\begin{pmatrix}\Delta_{34}C_0 I_{10} e^{-\mu T_1} + \\ +\frac{|p_2\gamma_1-q_2| I_{10} e^{-\mu T_1}}{\mu} + \\ +\frac{|p_2\gamma_2-q_2| I_{20} e^{-\mu T_2}}{\mu} + \\ +\frac{|p_2\gamma_1-q_2| I_{30}}{\mu} + \frac{|p_2\gamma_2-q_2| I_{40}}{\mu}\end{pmatrix} +$$
$$+\sum_{p=1}^{m}\begin{pmatrix}\left|g_p^1\right||q_2|\begin{pmatrix}|p_1| I_{10} e^{-\mu T_1} + \\ +|p_2| I_{20} e^{-\mu T_2} + \\ +|p_1| I_{30} + |p_2| I_{40}\end{pmatrix}^p + \\ +|p_2|\left|g_p^2\right|\begin{pmatrix}|q_1| I_{10} e^{-\mu T_1} + \\ +|q_2| I_{20} e^{-\mu T_2} + \\ +|q_1| I_{30} + |q_2| I_{40}\end{pmatrix}^p\end{pmatrix}\frac{e^{p\mu T_0}+\ldots+1}{p\mu} \le$$
$$\le I_{30} e^{\mu(t-kT_0)}$$

$$\left|B_k^{(4)}(I_1,I_2,I_3,I_4)(t)\right| \le$$
$$\le \frac{e^{\mu(t-kT_0)}}{\Delta_{34}C_0}\begin{pmatrix}\Delta_{34}C_0 I_{20} e^{-\mu T_2} + \\ +\frac{|p_1\gamma_1+q_1| I_{10} e^{-\mu T_1}}{\mu} + \\ +\frac{|p_1\gamma_2+q_1| I_{20} e^{-\mu T_2}}{\mu} + \\ +\frac{|p_1\gamma_1+q_1| I_{30}}{\mu} + \frac{|p_1\gamma_2+q_1| I_{40}}{\mu}\end{pmatrix} +$$
$$+\sum_{p=1}^{m}\begin{pmatrix}\left|g_p^2\right||p_1|\begin{pmatrix}|q_1| I_{10} e^{-\mu T_1} + \\ +|q_2| I_{20} e^{-\mu T_2} + \\ +|q_1| I_{30} + |q_2| I_{40}\end{pmatrix}^p + \\ +|q_1|\left|g_p^1\right|\begin{pmatrix}|p_1| I_{10} e^{-\mu T_1} + \\ +|p_2| I_{20} e^{-\mu T_2} + \\ +|p_1| I_{30} + |p_2| I_{40}\end{pmatrix}^p\end{pmatrix}\frac{e^{p\mu T_0}+\ldots+1}{p\mu} \le$$
$$\le I_{40} e^{\mu(t-kT_0)}$$

It remains to show that B is a contractive operator.

We notice that $T_2 > T_1 \Rightarrow -\mu T_2 < -\mu T_1 \Rightarrow e^{-\mu T_2} < e^{-\mu T_1}$.

Then

$$\max\begin{Bmatrix}|A_{11}| e^{-\mu T_1} + |A_{12}| e^{-\mu T_2}; |A_{21}| e^{-\mu T_1} + |A_{22}| e^{-\mu T_2}; \\ |B_{11}| e^{-\mu T_1} + |B_{12}| e^{-\mu T_2}; |B_{21}| e^{-\mu T_1} + |B_{22}| e^{-\mu T_2}\end{Bmatrix} \le$$
$$\le e^{-\mu T_1}\max\begin{Bmatrix}|A_{11}| + |A_{12}|; |A_{21}| + |A_{22}|; \\ |B_{11}| + |B_{12}|; |B_{21}| + |B_{22}|\end{Bmatrix} = K(\mu)$$

and for $t \in [kT_0, (k+1)T_0]$ we have

$$\left|B_k^{(1)}(I_1,I_2,I_3,I_4)(t)-B_k^{(1)}(\bar I_1,\bar I_2,\bar I_3,\bar I_4)(t)\right|\le$$
$$\le|A_{11}|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|+|A_{12}|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|\le$$
$$\le|A_{11}|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|e^{-\mu(t-kT_0-T_1)}e^{\mu(t-kT_0-T_1)}+$$
$$+|A_{12}|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|e^{-\mu(t-kT_0-T_2)}e^{\mu(t-kT_0-T_2)}\le$$
$$\le|A_{11}|e^{\mu(t-kT_0-T_1)}\rho_{j_1(k)}\left(I_3,\bar I_3\right)+$$
$$+|A_{12}|e^{\mu(t-kT_0-T_2)}\rho_{j_2(k)}\left(I_4,\bar I_4\right)\le$$
$$\le e^{\mu(t-kT_0)}\left(|A_{11}|e^{-\mu T_1}+|A_{12}|e^{-\mu T_2}\right)\times$$
$$\times\max\left\{\rho_{j_1(k)}\left(I_3,\bar I_3\right),\rho_{j_2(k)}\left(I_4,\bar I_4\right)\right\}$$

that implies

$$\rho_k\left(B_k^{(1)}(I_1,I_2,I_3,I_4),B_k^{(1)}(\bar I_1,\bar I_2,\bar I_3,\bar I_4)\right)\le$$
$$\le|A_{11}|e^{-\mu T_1}\rho_{j_1(k)}\left(I_3,\bar I_3\right)+|A_{12}|e^{-\mu T_2}\rho_{j_2(k)}\left(I_4,\bar I_4\right)\le$$
$$\le\left(|A_{11}|e^{-\mu T_1}+|A_{12}|e^{-\mu T_2}\right)\max\left\{\rho_{j_1(k)}\left(I_3,\bar I_3\right),\rho_{j_2(k)}\left(I_4,\bar I_4\right)\right\}\le$$
$$\le\left(|A_{11}|e^{-\mu T_1}+|A_{12}|e^{-\mu T_2}\right)\times$$
$$\times\max\begin{Bmatrix}\rho_{j_1(k)}\left(I_1,\bar I_1\right),\rho_{j_2(k)}\left(I_2,\bar I_2\right),\\ \rho_{j_1(k)}\left(I_3,\bar I_3\right),\rho_{j_2(k)}\left(I_4,\bar I_4\right)\end{Bmatrix}$$

$$\left|B_k^{(2)}(I_1,I_2,I_3,I_4)(t)-B_k^{(2)}(\bar I_1,\bar I_2,\bar I_3,\bar I_4)(t)\right|\le$$
$$\le|A_{21}|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|e^{-\mu(t-kT_0-T_1)}e^{\mu(t-kT_0-T_1)}+$$
$$+|A_{22}|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|e^{-\mu(t-kT_0-T_2)}e^{\mu(t-kT_0-T_2)}\le$$
$$\le e^{\mu(t-kT_0)}\left(|A_{21}|e^{-\mu T_1}+|A_{22}|e^{-\mu T_2}\right)\times$$
$$\times\max\left\{\rho_{j_1(k)}\left(I_3,\bar I_3\right),\rho_{j_2(k)}\left(I_4,\bar I_4\right)\right\}$$

which implies

$$\rho_k\left(B_k^{(2)}(I_1,I_2,I_3,I_4),B_k^{(2)}(\bar I_1,\bar I_2,\bar I_3,\bar I_4)\right)\le$$
$$\le|A_{21}|e^{-\mu T_1}\rho_{j_1(k)}\left(I_3,\bar I_3\right)+|A_{22}|e^{-\mu T_2}\rho_{j_2(k)}\left(I_4,\bar I_4\right)\le$$
$$\le\left(|A_{21}|e^{-\mu T_1}+|A_{22}|e^{-\mu T_2}\right)\max\left\{\rho_{j_1(k)}\left(I_3,\bar I_3\right),\rho_{j_2(k)}\left(I_4,\bar I_4\right)\right\}\le$$
$$\le\left(|A_{21}|e^{-\mu T_1}+|A_{22}|e^{-\mu T_2}\right)\times$$
$$\times\max\begin{Bmatrix}\rho_{j_1(k)}\left(I_1,\bar I_1\right),\rho_{j_2(k)}\left(I_2,\bar I_2\right),\\ \rho_{j_1(k)}\left(I_3,\bar I_3\right),\rho_{j_2(k)}\left(I_4,\bar I_4\right)\end{Bmatrix}$$

Further on, in view of

$$\frac{df_1(u)}{du}=\sum_{p=1}^{m}pg_p^{(1)}u^{p-1},\ \frac{df_2(u)}{du}=\sum_{p=1}^{m}pg_p^{(2)}u^{p-1}$$

$$\left|\frac{df_1(p_1I_1(t-T_1)+p_2I_2(t-T_2)-p_1I_3(t)-p_2I_4(t))}{du}\right|\le$$
$$\le\sum_{p=1}^{m}p\left|g_p^1\right|\begin{pmatrix}p_1I_{10}e^{-\mu T_1}+\\ +p_2I_{20}e^{-\mu T_2}+\\ +p_1I_{30}+p_2I_{40}\end{pmatrix}^{p-1}e^{p\mu(t-kT_0)}$$

$$\left|\frac{df_2\left(q_1I_1(t-T_1)+q_2I_2(t-T_2)-q_1I_3(t)-q_2I_4(t)\right)}{du}\right|\le$$
$$\le\sum_{p=1}^{m}p\left|g_p^2\right|\begin{pmatrix}q_1I_{10}e^{-\mu T_1}+\\ +q_2I_{20}e^{-\mu T_2}+\\ +q_1I_{30}+q_2I_{40}\end{pmatrix}^{p-1}e^{p\mu(t-kT_0)}$$

we have

$$\left|B_k^{(3)}(I_1,I_2,I_3,I_4)(t)-B_k^{(3)}(\bar I_1,\bar I_2,\bar I_3,\bar I_4)(t)\right|\le$$
$$\le\int_{T+kT_0}^{t}\left|U_3(I_1,I_2,I_3,I_4)(s)-U_3(\bar I_1,\bar I_2,\bar I_3,\bar I_4)(s)\right|ds+$$
$$+\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(U_3(I_1,I_2,I_3,I_4)(s)-U_3(\bar I_1,\bar I_2,\bar I_3,\bar I_4)\right)ds\right|=$$
$$=W_1+W_2$$

$$W_1\le e^{\mu(t-kT_0)}\begin{pmatrix}e^{-\mu T_1}+\dfrac{|p_2\gamma_1-q_2|\left(1+e^{-\mu T_1}\right)}{\mu\Delta_{34}C_0}+\\ +\dfrac{|p_2\gamma_2-q_2|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}\end{pmatrix}\times$$
$$\times\max\begin{Bmatrix}\rho^{j_1(k)}(I_1,\bar I_1),\rho^{j_2(k)}(I_2,\bar I_2),\\ \rho^{(k)}(I_3,\bar I_3),\rho^{(k)}(I_4,\bar I_4)\end{Bmatrix}+$$
$$+e^{\mu(t-kT_0)}\max\begin{Bmatrix}\rho^{j_1(k)}(I_1,\bar I_1),\rho^{j_2(k)}(I_2,\bar I_2),\\ \rho^{(k)}(I_3,\bar I_3),\rho^{(k)}(I_4,\bar I_4)\end{Bmatrix}\times$$
$$\times\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times$$
$$\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\begin{pmatrix}p_1I_{10}e^{-\mu T_1}+\\ +p_2I_{20}e^{-\mu T_2}+\\ +p_1I_{30}+p_2I_{40}\end{pmatrix}^{p-1}+$$

$$
+e^{\mu(t-kT_0)}\max\left\{\begin{matrix}\rho^{j_1(k)}(I_1,\bar{I}_1),\rho^{j_2(k)}(I_2,\bar{I}_2),\\ \rho^{(k)}(I_3,\bar{I}_3),\rho^{(k)}(I_4,\bar{I}_4)\end{matrix}\right\}\times
$$
$$
\times\frac{p_2\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times
$$
$$
\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(q_1 I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}\le
$$
$$
\le e^{\mu(t-kT_0)}\max\left\{\rho^{j_1(k)}(I_1,\bar{I}_1),\rho^{j_2(k)}(I_2,\bar{I}_2),\rho^{(k)}(I_3,\bar{I}_3),\rho^{(k)}(I_4,\bar{I}_4)\right\}
$$
$$
\times\Bigg[e^{-\mu T_1}+\frac{\left|p_2\gamma_1-q_2\right|\left(1+e^{-\mu T_1}\right)}{\mu\Delta_{34}C_0}+\frac{\left|p_2\gamma_2-q_2\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+
$$
$$
+\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(p_1 I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}+
$$
$$
+\frac{p_2\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(q_1 I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}\Bigg],
$$

$$
W_2\le\left(e^{\mu T_0}-1\right)\max\left\{\begin{matrix}\rho^{j_1(k)}(I_1,\bar{I}_1),\rho^{j_2(k)}(I_2,\bar{I}_2),\\ \rho^{(k)}(I_3,\bar{I}_3),\rho^{(k)}(I_4,\bar{I}_4)\end{matrix}\right\}\times
$$
$$
\times\Bigg[e^{-\mu T_1}+\frac{\left|p_2\gamma_1-q_2\right|\left(1+e^{-\mu T_1}\right)}{\mu\Delta_{34}C_0}+\frac{\left|p_2\gamma_2-q_2\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+
$$
$$
+\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times
$$
$$
\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(p_1 I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}+
$$
$$
+\frac{p_2\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(q_1 I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}\Bigg]
$$

Therefore

$$
\left|B_k^{(3)}(I_1,I_2,I_3,I_4)(t)-B_k^{(3)}(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4)(t)\right|\le
$$
$$
\le W_1+W_2\le
$$
$$
\le e^{\mu(t-kT_0)}\max\left\{\rho_{j_1(k)}(I_1,\bar{I}_1),\rho_{j_2(k)}(I_2,\bar{I}_2),\rho_k(I_3,\bar{I}_3),\rho_k(I_4,\bar{I}_4)\right\}
$$
$$
\times\Bigg[e^{-\mu T_1}+e^{\mu T_0}\frac{\left|p_2\gamma_1-q_2\right|\left(1+e^{-\mu T_1}\right)}{\mu\Delta_{34}C_0}+
$$
$$
+\frac{\left|p_2\gamma_2-q_2\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+
$$
$$
+e^{\mu T_0}\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times
$$
$$
\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(p_1 I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}+
$$
$$
+e^{\mu T_0}\frac{p_2\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times
$$
$$
\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(q_1 I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}\Bigg]
$$

that implies

$$
\rho_k\left(B_k^{(3)}(I_1,I_2,I_3,I_4),B_k^{(3)}(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4)\right)\le
$$
$$
\le\max\left\{\rho_{j_1(k)}(I_1,\bar{I}_1),\rho_{j_2(k)}(I_2,\bar{I}_2),\rho_k(I_3,\bar{I}_3),\rho_k(I_4,\bar{I}_4)\right\}\times
$$
$$
\times\Bigg[e^{-\mu T_1}+e^{\mu T_0}\frac{\left|p_2\gamma_1-q_2\right|\left(1+e^{-\mu T_1}\right)+\left|p_2\gamma_2-q_2\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+
$$
$$
+e^{\mu T_0}\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times
$$
$$
\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(p_1 I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}+
$$
$$
+e^{\mu T_0}\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times
$$
$$
\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times
$$
$$
\times\left(p_1 I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}+
$$

$$+e^{\mu T_0}\frac{p_2\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\left.\times\left(q_1\,I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}\right]$$

Finally, we have

$$\left|B_k^{(4)}(I_1,I_2,I_3,I_4)(t)-B_k^{(4)}(\overline{I}_1,\overline{I}_2,\overline{I}_3,\overline{I}_4)(t)\right|\le$$

$$\le\int_{T+kT_0}^{t}\left|U_4(I_1,I_2,I_3,I_4)(s)-U_4(\overline{I}_1,\overline{I}_2,\overline{I}_3,\overline{I}_4)(s)\right|ds+$$

$$+\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(U_4(I_1,I_2,I_3,I_4)(s)-U_4(\overline{I}_1,\overline{I}_2,\overline{I}_3,\overline{I}_4)\right)ds\right|=$$

$$=J_1+J_2;$$

$$J_1\le e^{\mu(t-kT_0)}\max\left\{\begin{matrix}\rho_{j_1(k)}(I_1,\overline{I}_1),\rho_{j_2(k)}(I_2,\overline{I}_2),\\ \rho_k(I_3,\overline{I}_3),\rho_k(I_4,\overline{I}_4)\end{matrix}\right\}\times$$

$$\times\left[e^{-\mu T_2}+\frac{\left|p_1\gamma_1+q_1\right|\left(1+e^{-\mu T_1}\right)}{\mu\Delta_{34}C_0}+\frac{\left|p_1\gamma_2+q_1\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+\right.$$

$$+\frac{p_1\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\times\left(q_1\,I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}+$$

$$+\frac{q_1\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\left.\times\left(p_1\,I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}\right]$$

$$J_2\le\left(e^{\mu T_0}-1\right)\max\left\{\begin{matrix}\rho_{j_1(k)}(I_1,\overline{I}_1),\rho_{j_2(k)}(I_2,\overline{I}_2),\\ \rho_k(I_3,\overline{I}_3),\rho_k(I_4,\overline{I}_4)\end{matrix}\right\}\times$$

$$\times\left[\frac{\left|p_1\gamma_1+q_1\right|\left(1+e^{-\mu T_1}\right)+\left|p_1\gamma_2+q_1\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+\right.$$

$$+\frac{p_1\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\times\left(q_1\,I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}+$$

$$+\frac{q_1\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\left.\times\left(p_1\,I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}\right]$$

Therefore

$$\left|B_k^{(4)}(I_1,I_2,I_3,I_4)(t)-B_k^{(4)}(\overline{I}_1,\overline{I}_2,\overline{I}_3,\overline{I}_4)(t)\right|\le$$

$$\le J_1+J_2\le$$

$$\le e^{\mu(t-kT_0)}\max\left\{\begin{matrix}\rho_{j_1(k)}(I_1,\overline{I}_1),\rho_{j_2(k)}(I_2,\overline{I}_2),\\ \rho_k(I_3,\overline{I}_3),\rho_k(I_4,\overline{I}_4)\end{matrix}\right\}\times$$

$$\times\left[e^{-\mu T_2}+e^{\mu T_0}\frac{\left|p_1\gamma_1+q_1\right|\left(1+e^{-\mu T_1}\right)+\left|p_1\gamma_2+q_1\right|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+\right.$$

$$+e^{\mu T_0}\frac{p_1\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\times\left(q_1\,I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}+$$

$$+e^{\mu T_0}\frac{q_1\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\left.\times\left(p_1\,I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}\right]$$

and then

$$\rho_k\left(B_k^{(4)}(I_1,I_2,I_3,I_4),B_k^{(4)}(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4)\right)\le$$

$$\le\max\left\{\rho_{j_1(k)}(I_1,\bar{I}_1),\rho_{j_2(k)}(I_2,\bar{I}_2),\rho_k(I_3,\bar{I}_3),\rho_k(I_4,\bar{I}_4)\right\}\times$$

$$\times\Bigg[e^{-\mu T_2}+e^{\mu T_0}\frac{|p_1\gamma_1+q_1|\left(1+e^{-\mu T_1}\right)+|p_1\gamma_2+q_1|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+$$

$$+e^{\mu T_0}\frac{p_1\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\times\left(q_1I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}+$$

$$+e^{\mu T_0}\frac{q_1\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\times$$

$$\times\sum_{p=1}^{m}\frac{p\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{p\mu}\times$$

$$\times\left(p_1I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}\Bigg]$$

Define the map

$$j\left(p_1,p_2,p_3,p_4\right):N_0\times N_0\times N_0\times N_0\to N_0\times N_0\times N_0\times N_0$$

in the following way:

$$j\left(k,k,k,k\right)=\begin{cases}\left(j_1(k),j_2(k),k,k\right), & \\ \quad\text{if }\ \rho_{j_1(k)}(I_3,\bar{I}_3)\le\rho_k(I_3,\bar{I}_3) & \\ \quad\text{and }\ \rho_{j_2(k)}(I_4,\bar{I}_4)\le\rho_k(I_4,\bar{I}_4) & \\ \left(j_1(k),j_2(k),j_1(k),j_2(k)\right), & \\ \quad\text{if }\ \rho_{j_1(k)}(I_3,\bar{I}_3)\ge\rho_k(I_3,\bar{I}_3) & \\ \quad\text{and }\ \rho_{j_2(k)}(I_4,\bar{I}_4)\ge\rho_k(I_4,\bar{I}_4) & \end{cases}$$

Therefore

$$\rho_{(k,k,k,k)}\left((B_k^{(1)},B_k^{(2)},B_k^{(3)},B_k^{(4)}),(\bar{B}_k^{(1)},\bar{B}_k^{(2)},\bar{B}_k^{(3)},\bar{B}_k^{(4)})\right)\le$$

$$\le K\rho_{j(k,k,k,k)}\left((I_1,I_2,I_3,I_4),(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4)\right)$$

where

$$K=\max\Big\{|A_{11}|e^{-\mu T_1}+|A_{12}|e^{-\mu T_2};|A_{21}|e^{-\mu T_1}+|A_{22}|e^{-\mu T_2}$$

$$+e^{-\mu T_1}+e^{\mu T_0}\frac{|p_2\gamma_1-q_2|\left(1+e^{-\mu T_1}\right)+|p_2\gamma_2-q_2|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+$$

$$+e^{\mu T_0}\frac{q_2\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{\mu}\times$$

$$\times\left(p_1I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}+$$

$$+e^{\mu T_0}\frac{p_2\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{\mu}\times$$

$$\times\left(q_1I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1};$$

$$e^{-\mu T_2}+e^{\mu T_0}\frac{|p_1\gamma_1+q_1|\left(1+e^{-\mu T_1}\right)+|p_1\gamma_2+q_1|\left(1+e^{-\mu T_2}\right)}{\mu\Delta_{34}C_0}+$$

$$+e^{\mu T_0}\frac{p_1\left(q_1e^{-\mu T_1}+q_2e^{-\mu T_2}+q_1+q_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{\left|g_p^2\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{\mu}\times$$

$$\times\left(q_1I_{10}e^{-\mu T_1}+q_2I_{20}e^{-\mu T_2}+q_1I_{30}+q_2I_{40}\right)^{p-1}+$$

$$+\frac{e^{\mu T_0}q_1\left(p_1e^{-\mu T_1}+p_2e^{-\mu T_2}+p_1+p_2\right)}{\Delta_{34}C_0}\sum_{p=1}^{m}\frac{\left|g_p^1\right|\left(e^{(p-1)\mu T_0}+\ldots+1\right)}{\mu}\times$$

$$\times\left(p_1I_{10}e^{-\mu T_1}+p_2I_{20}e^{-\mu T_2}+p_1I_{30}+p_2I_{40}\right)^{p-1}\Big\}<1$$

So, the operator B is contractive in the sense of definition from Chapter 1 in [3] and consequently has a unique fixed point. This fixed point is a periodic solution of (cf. equations (19)-(22) in [2]).

Theorem 1 is thus proved.

Finally, we note that the solution can be approximated by a sequence of successive approximations with advanced prescribed accuracy.

IV. Verification

Since our goal was to find

$$U_{NE}=u_2(0,t);\quad U_{FE}=u_2(\Lambda,t)$$

we have (cf. (3.3/11), (3.4/12))

$$u_2(0,t)=q_1I_1(0,t)+q_2I_2(0,t)-q_1I_3(0,t)-q_2I_4(0,t)=$$
$$=q_1I_1(t)+q_2I_2(t)-q_1I_3(t-T_1)-q_2I_4(t-T_2)$$

$$u_2(\Lambda,t)=q_1I_1(\Lambda,t)+q_2I_2(\Lambda,t)-q_1I_3(\Lambda,t)-q_2I_4(\Lambda,t)=$$
$$=q_1I_1(t-T_1)+q_2I_2(t-T_2)-q_1I_3(t)-q_2I_4(t)$$

where $\left(I_1(t),I_2(t),I_3(t),I_4(t)\right)$ is the solution obtained in the main theorem.

We have to check the conditions of our Theorem 1 using data from [5].

If the *IV* characteristic of the nonlinear resistive element is $f_1(u)=f_2(u)=-0.12u+0.8u^3$, the length of the line is □ = 1 m. The specific parameters are

$L_{11}=L_G=L_R=L_{22}=0.8529$ □H/m;

$L_m=0.3725$ μH/m; $L_{12}=L_{21}=L_m$;

$C_{11}=C_G+C_m=C_R+C_m=C_{22}=46.762$ pF/m

$C_{12}=C_{21}=-C_m=-18.036$ pF/m

Then $T=\Lambda\sqrt{L_GC_G}=\Lambda\sqrt{L_RC_R}=4.95\times10^{-9}$

Let us check the propagation of the waves with length $l_1=(1/4)10^{-3}$ and $l_2=(1/2)10^{-3}$. We have

$$f_1 = 1/\left(l_1\sqrt{L_G C_G}\right) = 1/\left((1/4)10^{-3}\times 4.95\times 10^{-9}\right) \approx 0.8\times 10^{12} \Rightarrow$$

$$T_1 = 1/\left(0.8\times 10^{12}\right) = 1.2\times 10^{-12}\ \text{sec};$$

$$f_2 = 1/\left(l_w\sqrt{L_G C_G}\right) = 1.6\times 10^{12} \Rightarrow T_2 = 0.625\times 10^{-12}\ \text{sec};$$

We choose $C_0 = 8\times10^{-12}$ F, $\mu = 10^{12}$, then $\mu\, T_0 = \mu_0 = 1$ and $T_1 = 2\times10^{-8}\times10^{12}\, T_0 = 20000\times T_0$. We also have $\mu\, C_0 = 10^{12}\times 8\times10^{-12} = 8$.

$$L_{12}C_{11} + L_{22}C_{12} = L_m\left(C_G + C_m\right) - L_R C_m =$$
$$= 0.3725\times 46.762 - 0.8529\times 18.036 = 2.036 \neq 0$$

$$L_{12}C_{22} + L_{11}C_{12} = L_m\left(C_R + C_m\right) - L_G C_m =$$
$$= 0.3725\times 46.762 - 0.8529\times 18.036 = 2.036 \neq 0$$

$$\Delta_C = C_{11}C_{22} - C_{12}^2 = \left(C_G + C_m\right)\left(C_R + C_m\right) - C_m^2 =$$
$$= 46.762^2 - \left(-18.036\right)^2 \approx 1861.3874 > 0$$

$$\Delta_L = L_G L_R - L_m^2 = L_{11}L_{22} - L_{12}^2 =$$
$$= 0.8529^{\,2} - 0.3725^{\,2} \approx 0.5887 > 0$$

$$\lambda_1 \approx \sqrt{0,0321157} \approx 0,1792\ ,\ \lambda_2 \approx \sqrt{0,0284} \approx 0,1686\ ,$$

$$L_{22} + \lambda_1.\lambda_2\Delta_L C_{11} \quad \approx 1,6846;$$

$$C_{22}\Delta_L + L_{11}\sqrt{\Delta_L\Delta_C} \quad \approx 55,7546;$$

$$\Delta_{12} = 0.427 + 24.051\times R_S R_{NE} + 3.5014\times R_{NE} + 19.6\times R_S\ ,$$

$$\Delta_{34} = -\left(0.427 + 24.051\times R_\Lambda R_{FE} + 3.5014\times R_{FE} + 19.6\times R_\Lambda\right)$$

$$\gamma_1 = \frac{1-\lambda_1^{\,2}\left(L_{11}C_{11} + L_{12}C_{12}\right)}{\lambda_1^{\,2}\left(L_{12}C_{11} + L_{22}C_{12}\right)} \approx -0.987979\,,$$

$$\gamma_2 = \frac{1-\lambda_2^{\,2}\left(L_{11}C_{11} + L_{12}C_{12}\right)}{\lambda_2^{\,2}\left(L_{12}C_{11} + L_{22}C_{12}\right)} \approx 1.038\,.$$

The inequalities from the main theorem are:

$$\frac{\left|\lambda_1 L_{12} + \left(L_{22}\lambda_1 + R_{NE}\right)\gamma_1\right|}{\left|\Delta_{12}\right|}\breve{U}_S =$$
$$= \frac{\left|0.0667 - 0.98798\times\left(0.1528 + R_{NE}\right)\right|}{0.427 + \left(24.051\times R_S + 3.5014\right)\times R_{NE} + 19.6\times R_S}\breve{U}_S \leq$$
$$\leq \min\left\{I_{10}, I_{20}\right\}$$

$$\frac{\left|\lambda_2 L_{12} + \left(L_{22}\lambda_2 + R_{NE}\right)\gamma_2\right|}{\left|\Delta_{12}\right|}\breve{U}_S =$$
$$= \frac{\left|0.0628 + 1.038\times\left(0.1437 + R_{NE}\right)\right|}{0.427 + \left(24.051\times R_S + 3.5014\right)\times R_{NE} + 19.6\times R_S}\breve{U}_S \leq$$
$$\leq \min\left\{I_{10}, I_{20}\right\}$$

We choose third order polynomial

$$f_1(u) = f_2(u) = -0.12u + 0.8u^3$$

It remains to show that the contractive conditions are satisfied. But this can be achieved by sufficiently large μ .

In view of

$p_1 = 0.1792\times(0.8529 - 0.3725\times0.987979)=0,0869,$

$p_2 = 0.1686\times(0.8529 + 0.3725\times1.038)=0,20898,$

$q_1 = 0.1792\times(0.3725 - 0.8529\times0.987979)= -0,08425$

$q_2 = 0.1686\times(0.3725 + 0.8529\times1.038)= 0,2120$

we can compute

$$\frac{\left|p_2\gamma_1 - q_2\right| + \left|p_2\gamma_2 - q_2\right|}{\Delta_{34}C_0}, \frac{q_2\left(p_1 + p_2\right)}{\Delta_{34}C_0}, \frac{q_2\left(p_1 + p_2\right)}{\Delta_{34}C_0},$$

$$\frac{\left|p_1\gamma_1 + q_1\right| + \left|p_1\gamma_2 + q_1\right|}{\mu\Delta_{34}C_0}, \frac{p_1\left(q_1 + q_2\right)}{\Delta_{34}C_0}, \frac{q_1\left(p_1 + p_2\right)}{\Delta_{34}C_0}.$$

V. Conclusion

The 3-conductor transmission line models an EMC problem. This problem leads to the mathematical formulation of a mixed problem for hyperbolic system of partial differential equations. We succeeded to reduce the mixed problem for the 3-conductor transmission line to an initial value problem on the boundary in part 1 of our study [2]. In this paper, we solve the system obtained consisting of two functional equations and two neutral equations with two different delays for the unknown functions I_1, I_2, I_3, I_4. In result, using our approach, we may find the currents and voltages of the 3-conductor transmission line.

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References

[1] Clayton R. Paul, "*Introduction to Electromagnetic Compatibility*", 2nd Edition, John Wiley & Sons (2006). ISBN 978-0-471-75500-5

[2] G. Angelov, "Analysis of a 3-Conductor Transmission Line with Nonlinear Resistive Loads," *2021 12th National Conference with International Participation (ELECTRONICA)*, 2021, pp. 1-6, DOI: 10.1109/ELECTRONICA52725.2021.9513679.

[3] V. G. Angelov, "*A Method for Analysis of Transmission Lines Terminated by Nonlinear Loads*", Nova Science, New York, 2014.

[4] G. Angelov, "Note on an Generalized Solution of the Three-Conductor Transmission Line Equations," *2020 International Conference on Mathematics and Computers in Science and Engineering (MACISE)*, 2020, pp. 278-283, doi: 10.1109/MACISE49704.2020.00058.

[5] C. R. Paul, "*Analysis of multi conductor transmission lines*", New York, A Willy Inter Science Publications, J. Willey & Sons, 1994.

[6]