Finding an Approximate Solution for a Non-Autonomous Dynamical System by Direct Using an Asymptotic Method to the Energy Expression

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Abstract. A method for finding periodic solutions of a dynamic system subjected to external periodic influence is presented. The solution is developed in series by a small parameter. Expressions for the energy in the different approximations are obtained. Non-resonant and resonant cases are considered. The solution in the first approximation is constructed as a power function of the solution in the zero approximation.

INTRODUCTION

Time-dependent dynamic systems are non-autonomous. Dynamic systems are considered, the time-independent part of which may have periodic behavior. The addition of external periodic perturbation also gives periodic decisions. Poincaré [1] made a major contribution to the existence and nature of such decisions. The theory developed by Poincaré is at the heart of the science of Theory of Oscillation [2]. Usually, it is impossible to find the exact solution for such systems. Therefore, approximate methods that use a small parameter (relative to 1) are applied. This parameter participates in the mathematical structure of the system itself or is related to the initial conditions [3]. In general, the resulting series are not convergent, but for a period of time that can be estimated, give a good approximation. The presence of external periodic perturbations can also lead to resonance in the system. In terms of its nature, two types are possible: powerful and parametric. Due to the way of setting the energy in this work, only the case of powerful resonance is considered.

DECOMPOSITION OF ENERGY IN SERIES BY SMALL PARAMETER. NONRESONANT MODE

Let the energy of the non-autonomous system be given in the form:

$$E = \frac{\omega^2 x^2 + \dot{x}^2}{2} - \varepsilon \omega^2 \frac{x^4}{4} - F \int \cos pt \, \dot{x} dt, \tag{1}$$

where F and ω^2 are constants, and ε is a small parameter.

Let:

$$pt = \tau. \tag{2}$$

For the energy expression (1) is obtained:

$$E = \frac{\omega^2 x^2 + p^2 x'^2}{2} - \varepsilon\omega^2 \frac{x^4}{4} - F \int \cos\tau\, x' d\tau, \tag{3}$$

where x' is the derivative with respect to τ.

x is obtained by development in series by the powers of the small parameter:

$$x = x_0 + \varepsilon x_1 + \cdots,. \tag{4}$$

$$x_0 = B\, cos\,\tau\,, B = const \tag{5}$$

$$E = E_0 + \varepsilon E_1 + \cdots. \tag{6}$$

It is easy to see that the members of the series (4) can be formally obtained by differentiation with respect to the small parameter:

$$\begin{gathered} x|_{\varepsilon=0} = x_0, \\ \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = x_\varepsilon|_{\varepsilon=0} = x_1, \\ \left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} = x_{\varepsilon\varepsilon}|_{\varepsilon=0} = 2x_{2,} \\ \dots\dots\dots\dots\dots\dots \\ \left.\frac{d^nx}{d\varepsilon^n}\right|_{\varepsilon=0} = x_{\varepsilon^n}|_{\varepsilon=0} = n!\, x_{n.} \end{gathered} \tag{7}$$

Thus, formally, the expressions for the different approximations can be obtained by differentiating the expression for the energy by the powers of the small parameter:

In the zero approximation, from the expression (3), it is obtain:

$$E_0 = \frac{\omega^2 {x_0}^2 + p^2 {x_0}'^2}{2} - \frac{F}{B}\int x_0\, dx_0. \tag{8}$$

Having in mind the relation:

$${x_0}'^2 = B^2 - {x_0}^2, \tag{9}$$

the expression is easily obtained:

$$2E_0 = p^2B^2 + \left(\omega^2 - p^2 - \frac{F}{B}\right){x_0}^2. \tag{10}$$

The expression to the right of equality (10) is a constant when the expression in the brackets becomes equal to 0. Thus, the quantities B and E_0 are finally determined:

$$B = \frac{F}{\omega^2 - p^2}, \tag{11}$$

$$E_0 = \frac{p^2F^2}{2(\omega^2 - p^2)^2}. \tag{12}$$

For the first approximation, the formula (13) is obtained:

$$E_1 = p^2 {x_0}' {x_1}' + \omega^2 x_0 x_1 - \omega^2 \frac{{x_0}^4}{4} - \frac{F}{B}\int x_0\, x_1' d\tau. \tag{13}$$

The x_1 function will be in the form:

$$x_1 = \mu_1 {x_0}^3 + \mu_2 x_0, \tag{14}$$

where μ_1 and μ_2 are constants to be determined. The first derivative of the function with respect to τ is also determined:

$$x_1' = 3\mu_1 {x_0}^2 x_0' + \mu_2 x_0'. \tag{15}$$

Substitute x_1 and x_1' in the expression for E_1 and using the relation:

$${x_0}'^2 = B^2 - {x_0}^2, \tag{16}$$

the expression (17) is obtained:

$$E_1 = \left(-3\mu_1 p^2 + \omega^2 \mu_1 - \frac{\omega^2}{4} - \frac{3F}{4B}\mu_1\right) {x_0}^4 + \left(3\mu_1 p^2 B^2 - p^2 \mu_2 + \omega^2 \mu_2 - \frac{F}{2B}\mu_2\right) {x_0}^2 + \mu_2 p^2 B^2. \tag{17}$$

In order for E_1 to be a constant, the expressions in the brackets must be equal to 0. Then:

$$\mu_1 = \frac{\omega^2}{-9p^2 + \omega^2}, \tag{18}$$

$$\mu_2 = -\frac{6p^2\omega^2}{-9p^2 + \omega^2} \cdot \frac{F^2}{(\omega^2 - p^2)^3}, \tag{19}$$

$$E_1 = -\frac{6p^4\omega^2}{-9p^2 + \omega^2} \cdot \frac{F^4}{(\omega^2 - p^2)^5}. \tag{20}$$

Expressions (18), (19) and (20) for the first approximation are applicable when $9p^2 - \omega^2$ and $\omega^2 - p^2$ are large enough in absolute value, i.e., the system is far from resonant. The use of larger approximations will lead to the possibility of obtaining other types of resonances, so in principle this approach excludes between the frequencies there are relationships of the type:

$$np + m\omega \sim 0;\ n, m \in \mathbb{Z}.$$

RESONANCE MODE

Assume that the natural frequency is close to that of the periodic frequency:

$$\omega = p + \varepsilon p p_1. \tag{21}$$

Let also the energy be expressed by the formula:

$$E = \frac{\omega^2 x^2 + \dot{x}^2}{2} - \varepsilon \omega^2 \frac{x^4}{4} - \varepsilon F_0 \int cos\, pt\, \dot{x} dt. \tag{22}$$

Let:

$$pt = \tau. \tag{23}$$

The expression for the energy (22) takes the form:

$$E = \frac{p^2 x^2 + p^2 x'^2}{2} + \varepsilon p p_1 x^2 - \varepsilon p^2 \frac{x^4}{4} - \varepsilon F_0 \int \cos\tau\, x' d\tau. \tag{24}$$

The zero approximation is:

$$x_0 = B\, cos\, \tau\, , B = const. \tag{25}$$

The solution is sought has the form:

$$x = x_0 + \varepsilon x_1, \tag{26}$$

$$E = E_0 + \varepsilon E_1. \tag{27}$$

For the energy in zero approximation:

$$E_0 = \frac{p^2 B^2}{2}. \tag{28}$$

The energy in the first approximation satisfies the equation:

$$E_1 = p^2 x_0{}' x_1{}' + p^2 x_0 x_1 + p p_1 x^2 - p^2 \frac{x_0{}^4}{4} - \frac{F}{B} \int x_0 \, dx_0. \tag{29}$$

The function x_1 has a form:

$$x_1 = \lambda x_0{}^3, \lambda = const. \tag{30}$$

Substituting x_1 in the formula for E_1 yields:

$$E_1 = 0, \tag{31}$$

$$\lambda = -\frac{1}{8}, \tag{32}$$

$$-\frac{3}{8} p^2 B^3 + p p_1 B - \frac{F}{2} = 0. \tag{33}$$

From the equation (33), the relation between the quantities p_1, B and F can be investigated. Such an analysis has been made, for example, in [4].

CONCLUSION

The aim of the present work is to demonstrate how the approximate periodic solution of a non-autonomous system can be found by directly applying a formal asymptotic decomposition to the expression for energy. The aim of the present work is to demonstrate how the approximate periodic solution of a non-autonomous system can be found by directly applying a formal asymptotic decomposition to the expression for energy. The results obtained by the proposed method completely coincide with the results obtained by the asymptotic method applied to the differential equation describing the dynamical system [2,3,5]. Briefly, the steps that apply are: 1) decomposition of the expression for energy by differentiation with respect to the small parameter; 2) finding the solution of the zero approximation; 3) finding the next approximations, looking in the form of a power function of the zero approximation, in the case when the small parameter participates as a multiplier in the power function in the basic equation for energy.

Calculations have been made to the first approximation, and according to the indicated scheme, they can be continued.

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