Etudes on combinatorial number theory

Miroslav Stoenchev[1], Venelin Todorov[2,3]

Technical University of Sofia, 8 Kliment Ohridski blvd., 1000 Sofia, Bulgaria[1]
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Department of Information Modeling, Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria[2]
Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, Department of Parallel Algorithms, Acad. G. Bonchev Str., Block 25 A, 1113, Sofia, Bulgaria[3]
mrs@tu-sofia.bg, vtodorov@math.bas.bg, venelin@parallel.bas.bg

Abstract. The purpose of this article is to consider a special class of combinatorial problems, the solution of which is realized by constructing finite sequences of ± 1. For example, for fixed $p \in \mathbb{N}$, is well known the existence of $n_p \in \mathbb{N}$ with the property: any set of n_p consecutive natural numbers can be divided into 2 sets, with equal sums of its p^{th}-powers. The considered property remains valid also for sets of finite arithmetic progressions of integers, real or complex numbers. The main observation here is the generalization of the results for arithmetic progressions with elements of complex field $\mathbb{C}$ to elements of arbitrary associative, commutative algebra.

1 Morse sequence

For every positive integer m, let us denote with $\vartheta(m)$ and $\varrho(m)$ respectively the number of occurences of digit 1 in the binary representation of m, and the position of first digit 1 in the binary representation of m. The Morse sequence $\{a_m\}_{m=1}^{\infty}$ is defined by

$$a_m = (-1)^{\vartheta(m)+\varrho(m)-2}.$$

The following properties are derived directly:

$$a_{2^k} = (-1)^k \quad \text{and} \quad a_{2^k+l} = -a_l \quad \text{for} \quad l = 1, 2 \ldots, 2^k; \; k \in \mathbb{N}.$$

The problem of finding a number n_p, such that the set $A_{n_p} = \{1, 2, \ldots, n_p\}$ is represented as disjoint union of two subsets, say B and C, with the property:

$$\sum_{b \in B} b^p = \sum_{c \in C} c^p,$$

is solved by the sequence $\{a_m\}_{m=1}^{\infty}$. Elementary proof is given below[1] and $n_p = 2^{p+1}$ has the desired property, with

$$B = \{m \in A_{n_p} : \; a_m = 1\}, \quad C = A_{n_p} \backslash B = \{m \in A_{n_p} : \; a_m = -1\}.$$

[1] Similar solutions and generalizations of the Prouhet-Tarry-Escot problem are considered in $[2, 3, 5, 6, 8]$

This result can be generalized to arbitrary arithmetic progressions of complex numbers. As example, if $a, d \in \mathbb{C},\ d \neq 0$ and $A_{n_p} = \{a+kd :\ k = 0, 1, \ldots, n_p-1\}$, then $n_p = 2^{p+1}$ and $B = \{a + kd \in A_{n_p} :\ a_{k+1} = 1\}$.

2 Formulation of the main results

Let us define $\{H_{n,m}(z)\}_{n,m=1}^{\infty}$, by

$$H_{n,m}(z) = \sum_{l=n}^{\infty} \sum_{k=1}^{2^l} a_k \left(P(z) + k.Q(z)\right)^m ,$$

where $P, Q \in \mathbb{C}[z]$ are complex polynomials.

Proposition 1 *If $m = 0, 1, \ldots, n-1$ then $H_{n,m} \equiv 0$, while if $n = 2t$ the following equality is satisfied*

$$H_{n,n}(z) = n!2^{\frac{n^2-n}{2}} Q^n(z).$$

Proposition 2 *Let $n \in \mathbb{N}$ be a even number and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are complex numbers, then*

$$\sum_{k=1}^{2^n} a_k(\alpha_1 + k)(\alpha_2 + k)\cdots(\alpha_n + k) = n!2^{\frac{n^2-n}{2}}.$$

Proposition 3 *If $P \in \mathbb{C}[z]$ is a complex polynomial, then*

$$\sum_{k=1}^{2^{1+\deg P}} a_k P(k) = 0.$$

Proposition 4 *Let p and k be positive integers. Then there exist $n \in \mathbb{N},\ n \leq 2^{p\lceil \log_2 k\rceil}$ and distinct square-free positive integers $x_{ij},\ i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n$ with the property:*

$$\sum_{j=1}^{n} x_{1j}^r = \sum_{j=1}^{n} x_{2j}^r = \cdots = \sum_{j=1}^{n} x_{kj}^r,\ \forall r = 1, 2, \ldots, p.$$

Proposition 5 *Let n and m be positve integers. Then there exists an integer $s = s(n, m)$ with the property: every $s-$element subset of Γ_n, where k runs through s consequtive integers, can be represented as disjoint union of m subsets, with equal sums of the elements in each one.*

The proof of each of the formulated above propositions[2], with the exception for 5, is based on the following lemma:

[2] This paper is continuation of [7], and contains complete proofs of the formulated propositions

Lemma 1. *Set $a, d \in \mathbb{C},\ d \neq 0,\ p \in \mathbb{N}$ and $A_{2^{p+1}} = \{a+kd :\ k = 0, 1, \ldots, 2^{p+1}-1\}$. Then there are sets $B \cap C = \emptyset,\ B \cup C = A_{2^{p+1}}$ such that*

$$\sum_{b\in B} b^p = \sum_{c\in C} c^p.$$

Corollary 1 *Under assumptions of lemma 1, it holds*

$$\sum_{b\in B} b^r = \sum_{c\in C} c^r,\ \ r = 0, 1, \ldots, p.$$

To prove the lemma 1 and its consequence, we define a sequence of polynomials: $\{T_{s,p}(z)\}_{s=0}^{\infty}$, through which we will gradually calculate the differences between the sums of equal powers of the elements in $B = \{a + kd \in A_{2^{p+1}} :\ a_{k+1} = 1\}$ and $C = \{a + kd \in A_{2^{p+1}} :\ a_{k+1} = -1\}$. For $s \geq 0$ set

$$T_{s,p}(z) = \sum_{k=0}^{4^{s+1}-1} a_{k+1}(z + kd)^p$$

and we calculate

$$T_{s,p}(z) = \sum_{0\leq k\leq 4^{s+1}-1;\ a_{k+1}=1} (z + kd)^p - \sum_{0\leq k\leq 4^{s+1}-1;\ a_{k+1}=-1} (z + kd)^p.$$

When $s \leq \frac{p-1}{2}$, set $z = a$ to obtain

$$T_{s,p}(a) = \sum_{b\leq a+(4^{s+1}-1)d} b^p - \sum_{c\leq a+(4^{s+1}-1)d} c^p,$$

where summation is by $b \in B,\ c \in C$.
Set $p = 2m + r,\ r \in \{0, 1\}$. Here and evewhere below the summations are performed on all $b \in B$ and $c \in C$, which satisfy the corresponding inequalities. When $r = 1$ we obtain

$$T_{m,p}(a) = \sum_{b\leq a+(4^{m+1}-1)d} b^p - \sum_{c\leq a+(4^{m+1}-1)d} c^p =$$

$$\sum_{b\leq a+(2^{p+1}-1)d} b^p - \sum_{c\leq a+(2^{p+1}-1)d} c^p = \sum_{b\in B} b^p - \sum_{c\in C} c^p.$$

When $r = 0$:

$$T_{m-1,p}(a) = \sum_{b\leq a+(2^{2m}-1)d} b^p - \sum_{c\leq a+(2^{2m}-1)d} c^p = \sum_{b\leq a+(2^p-1)d} b^p - \sum_{c\leq a+(2^p-1)d} c^p;$$

On the other hand

$$\sum_{a+2^pd\leq b\leq a+(2^{p+1}-1)d} b^p - \sum_{a+2^pd\leq c\leq a+(2^p-1)d} c^p = \sum_{2^p\leq m\leq 2^{p+1}-1} a_{m+1}(a + md)^p =$$

$$= \sum_{k=0}^{2^p-1} a_{2^p+k+1}\left(a+(2^p+k)d\right) = -\sum_{k=0}^{2^p-1} a_{k+1}\left(a+(2^p+k)d\right) =$$

$$= -\sum_{k=0}^{2^p-1} a_{k+1}\left((a+2^pd)+kd\right) = -\sum_{k=0}^{2^{2m}-1} a_{k+1}\left((a+2^pd)+kd\right) =$$

$$= -T_{m-1,p}(a+2^pd).$$

Therefore, for $p = 2m$ we obtain

$$\sum_{b\in B} b^p - \sum_{c\in C} c^p = T_{m-1,p}(a) - T_{m-1,p}(a+2^pd).$$

Summarized:

$$\sum_{b\in B} b^p - \sum_{c\in C} c^p = \begin{cases} T_{m,p}(a), \text{ for } p = 2m+1 \\ T_{m-1,p}(a) - T_{m-1,p}(a+2^pd), \text{ for } p = 2m \end{cases}$$

3 Proof of the main results

Lemma 1 follows directly from :

Proposition 6

$$T_{m-1,p}(z) = \begin{cases} 0, \textit{ for } p = 2m-1 \\ p!2^{\frac{p^2-p}{2}} d^p, \textit{ for } p = 2m \end{cases}$$

Proof 1 *Let us determine the polynomials* $\{T_{s,p}(z)\}_{s=0}^{\infty}$ *by finding recurrent formula. Since* $a_1 = a_4 = 1,\ a_2 = a_3 = -1,$ *then*

$$T_{0,p}(z) = (z+3d)^p - (z+2d)^p - (z+d)^p + z^p$$

We will prove that for all $s \geq 1$ *is valid*

$$T_{s,p}(z) = T_{s-1,p}(z+3.4^sd) - T_{s-1,p}(z+2.4^sd) - T_{s-1,p}(z+4^sd) + T_{s-1,p}(z)$$

For example, if $s = 1$ *then:*

$$T_{1,p}(z) = \sum_{k=0}^{15} a_{k+1}(z+kd)^p = \sum_{k=0}^{3} a_{k+1}(z+kd)^p + \sum_{k=4}^{7} a_{k+1}(z+kd)^p +$$

$$+\sum_{k=8}^{11} a_{k+1}(z+kd)^p + \sum_{k=12}^{15} a_{k+1}(z+kd)^p = T_{0,p}(z) + \sum_{m=0}^{3} a_{2^2+m+1}((z+4d)+md)^p +$$

$$+\sum_{m=0}^{3} a_{2^3+m+1}((z+2.4d)+md)^p + \sum_{m=0}^{3} a_{2^3+2^2+m+1}((z+3.4d)+md)^p =$$

$$= T_{0,p}(z) - \sum_{m=0}^{3} a_{m+1}((z+4d)+md)^p - \sum_{m=0}^{3} a_{m+1}((z+2.4d)+md)^p +$$

$$+ \sum_{m=0}^{3} a_{m+1}((z+3.4d)+md)^p = T_{0,p}(z) - T_{0,p}(z+4d) - T_{0,p}(z+2.4d) + T_{0,p}(z+3.4d)$$

The proof is similar in the general case:

$$T_{s,p}(z) = \sum_{k=0}^{4^{s+1}-1} a_{k+1}(z+kd)^p = \sum_{k=0}^{4^s-1} a_{k+1}(z+kd)^p + \sum_{k=4^s}^{2.4^s-1} a_{k+1}(z+kd)^p +$$

$$+ \sum_{k=2.4^s}^{3.4^s-1} a_{k+1}(z+kd)^p + \sum_{k=3.4^s}^{4^{s+1}-1} a_{k+1}(z+kd)^p =$$

$$= T_{s-1,p}(z) + \sum_{m=0}^{4^s-1} a_{4^s+m+1}((z+4^s d)+md)^p +$$

$$+ \sum_{m=0}^{4^s-1} a_{2.4^s+m+1}((z+2.4^s d)+md)^p + \sum_{m=0}^{4^s-1} a_{3.4^s+m+1}((z+3.4^s d)+md)^p =$$

$$= T_{s-1,p}(z+3.4^s d) - T_{s-1,p}(z+2.4^s d) - T_{s-1,p}(z+4^s d) + T_{s-1,p}(z),$$

whereby the necessary recurrent formula is established.

In the case $1 \le s \le \left[\frac{p}{2}\right] - 1$, *we prove that* $T_{s,p}(z)$ *has the type:*

$$T_{s,p}(z) =$$

$$= \sum_{i_1=2s}^{p-2} \sum_{i_2=2(s-1)}^{i_1-2} \sum_{i_3=2(s-2)}^{i_2-2} \cdots \sum_{i_{s+1}=0}^{i_s-2} \binom{p}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_s}{i_{s+1}} d^{p-i_{s+1}} L_{s,p} z^{i_{s+1}},$$

where

$$L_{s,p} =$$

$$= (3^{p-i_1}-2^{p-i_1}-1)(3^{i_1-i_2}-2^{i_1-i_2}-1)\ldots(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)4^{i_1+i_2+\cdots+i_s-si_{s+1}}$$

Indeed, when $s = 0$ *follows:*

$$T_{0,p}(z) = (z+3d)^p - (z+2d)^p - (z+d)^p + z^p =$$

$$\sum_{i_1=0}^{p-2} \binom{p}{i_1} (3^{p-i_1} - 2^{p-i_1} - 1) d^{p-i_1} z^{i_1};$$

$$T_{1,p}(z) = T_{0,p}(z) - T_{0,p}(z+4d) - T_{0,p}(z+2.4d) + T_{0,p}(z+3.4d) =$$

$$= \sum_{i_1=0}^{p-2} \binom{p}{i_1} (3^{p-i_1}-2^{p-i_1}-1) d^{p-i_1} \left((z+12d)^{i_1} - (z+8d)^{i_1} - (z+4d)^{i_1} + z^{i_1}\right) =$$

$$=\sum_{i_1=2}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}\left((z+12d)^{i_1}-(z+8d)^{i_1}-(z+4d)^{i_1}+z^{i_1}\right)=$$

$$=\textstyle\sum_{i_1=2}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}\left(z^{i_1}+\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}(3^{i_1-i_2}-2^{i_1-i_2}-1)(4d)^{i_1-i_2}z^{i_2}\right)=$$

$$=\textstyle\sum_{i_1=2}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}\left(\sum_{i_2=0}^{i_1-2}\binom{i_1}{i_2}(3^{i_1-i_2}-2^{i_1-i_2}-1)(4d)^{i_1-i_2}z^{i_2}\right)=$$

$$=\sum_{i_1=2}^{p-2}\sum_{i_2=0}^{i_1-2}\binom{p}{i_1}\binom{i_1}{i_2}(3^{p-i_1}-2^{p-i_1}-1)(3^{i_1-i_2}-2^{i_1-i_2}-1)4^{i_1-i_2}d^{p-i_2}z^{i_2}=$$

$$=\sum_{i_1=2}^{p-2}\sum_{i_2=0}^{i_1-2}\binom{p}{i_1}\binom{i_1}{i_2}d^{p-i_2}L_{1,p}z^{i_2},$$

whereby the assertion is established for $s=1$.*Suppose that for some* $s\geq 2,\ T_{s-1,p}(z)$ *satisfies the recurrent formula and denote*

$$G^{s,p}_{i_1,i_2,\dots,i_{s+1}}=\binom{p}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_s}{i_{s+1}}d^{p-i_{s+1}}L_{s,p},\ \ \textit{when}\ s\geq 1.$$

Direct calculation shows:

$$T_{s,p}(z)=T_{s-1,p}(z+3.4^sd)-T_{s-1,p}(z+2.4^sd)-T_{s-1,p}(z+4^sd)+T_{s-1,p}(z)=$$

$$=\sum_{i_1=2(s-1)}^{p-2}\ \sum_{i_2=2(s-2)}^{i_1-2}\ \sum_{i_3=2(s-3)}^{i_2-2}\cdots\sum_{i_s=0}^{i_{s-1}-2}G^{s-1,p}_{i_1,i_2,\dots,i_s}\times$$

$$\times\left((z+3.4^sd)^{i_s}-(z+2.4^sd)^{i_s}-(z+4^sd)^{i_s}+z^{i_s}\right)=$$

$$=\sum_{i_1=2(s-1)}^{p-2}\ \sum_{i_2=2(s-2)}^{i_1-2}\cdots\sum_{i_s=0}^{i_{s-1}-2}G^{s-1,p}_{i_1,i_2,\dots,i_s}\times$$

$$\times\left(z^{i_s}+\sum_{i_{s+1}=0}^{i_s}\binom{i_s}{i_{s+1}}(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)4^{s(i_s-i_{s+1})}d^{i_s-i_{s+1}}z^{i_{s+1}}\right)=$$

$$=\sum_{i_1=2s}^{p-2}\ \sum_{i_2=2(s-1)}^{i_1-2}\cdots\sum_{i_s=2}^{i_{s-1}-2}G^{s-1,p}_{i_1,i_2,\dots,i_s}\times$$

$$\times\sum_{i_{s+1}=0}^{i_s-2}\binom{i_s}{i_{s+1}}(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)4^{s(i_s-i_{s+1})}d^{i_s-i_{s+1}}z^{i_{s+1}}=$$

$$=\sum_{i_1=2s}^{p-2}\ \sum_{i_2=2(s-1)}^{i_1-2}\cdots\sum_{i_s=2}^{i_{s-1}-2}\ \sum_{i_{s+1}=0}^{i_s-2}G^{s-1,p}_{i_1,i_2,\dots,i_s}\binom{i_s}{i_{s+1}}\times$$

$$\times\left(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1\right)4^{s(i_s-i_{s+1})}d^{i_s-i_{s+1}}z^{i_{s+1}}=$$

$$= \sum_{i_1=2s}^{p-2} \sum_{i_2=2(s-1)}^{i_1-2} \cdots \sum_{i_s=2}^{i_{s-1}-2} \sum_{i_{s+1}=0}^{i_s-2} G^{s,p}_{i_1,i_2,\ldots,i_{s+1}} z^{i_{s+1}},$$

which prove that $T_{s,p}(z)$ satisfies the recurrent formula.
Let us determine the degree of $T_{s,p}(z),\ s \geq 0$. According to the derived formula we find $i_{s+1} \leq i_s-2 \leq i_{s-1}-4 \leq \cdots \leq i_1-2s \leq p-2(s+1)$, as equality is reached everywhere. Therefore $\deg T_{s,p}(z) = p-2(s+1)$. If $p = 2m+r,\ r \in \{0,1\}$, then

$$\deg T_{m-1,p}(z) = p - 2m = r.$$

For $r = 0$ we obtain that $T_{m-1,p}(z)$ is a constant, equal to $p!2^{\frac{p^2-p}{2}} d^p$. Indeed

$$T_{m-1,p}(z) = \sum_{i_1=2(m-1)}^{p-2} \sum_{i_2=2(m-2)}^{i_1-2} \cdots \sum_{i_{m-1}=2}^{i_{m-2}-2} \sum_{i_m=0}^{i_{m-1}-2} G^{m-1,p}_{i_1,i_2,\ldots,i_{s+1}} z^{i_m} =$$

$$= \sum_{i_1=2(m-1)}^{2(m-1)} \sum_{i_2=2(m-2)}^{2(m-2)} \cdots \sum_{i_{m-1}=2}^{2} \sum_{i_m=0}^{0} G^{m-1,p}_{i_1,i_2,\ldots,i_{s+1}} z^{i_m} =$$

$$= G^{m-1,p}_{p-2,p-4,p-6\ldots,2,0} = \binom{p}{p-2}\binom{p-2}{p-4}\cdots\binom{4}{2}\binom{2}{0} d^p L_{m-1,p} = \frac{p!d^p}{2^m} L_{m-1,p} =$$

$$= p!2^{\frac{p^2-p}{2}} d^p \Longrightarrow T_{m-1,p}(z) = p!2^{\frac{p^2-p}{2}} d^p,\ \text{when } p = 2m.$$

In the case $r = 1$, we will prove that $T_{m,p}(z) = 0$:

$$T_{m,p}(z) = T_{m-1,p}(z+3.4^m d) - T_{m-1,p}(z+2.4^m d) -$$
$$T_{m-1,p}(z+4^m d) + T_{m-1,p}(z) =$$
$$\sum_{i_1=2(m-1)}^{p-2} \sum_{i_2=2(m-2)}^{i_1-2} \cdots \sum_{i_m=0}^{i_{m-1}-2} G^{m-1,p}_{i_1,i_2,\ldots,i_m}$$
$$((z+3.4^m d)^{i_m} - (z+2.4^m d)^{i_m} - (z+4^m d)^{i_m} + z^{i_m}) = 0,$$

the last equations is valid, since the summation index i_m takes values 0 and 1. Thus the proposition 6 is proved. □

3.1 Proof of corollary 1

Proof 2 *According to 6 for every $m \in \mathbb{N}$ is valid $T_{m,2m+1}(z) \equiv 0$ and $T_{m-1,2m}(z) = (2m)!2^{2m^2-m} d^{2m}$. Then $T_{m,2m}(z) \equiv 0$, due to*

$$T_{m,2m}(z) = T_{m-1,2m}(z+3.4^m d) - T_{m-1,2m}(z+2.4^m d) -$$

$$-T_{m-1,2m}(z+4^m d) + T_{m-1,2m}(z) = 0.$$

Using the recurrent formula we obtain:

$$T_{s,2k}(z) = T_{s,2k+1}(z) \equiv 0,\ \textit{for all } s \geq k,\ \textit{i.e.}$$

$$T_{s,k}(z) \equiv 0,\ \textit{for all } s \geq \left[\frac{k}{2}\right].$$

There are two cases, first case: $p = 2m+1$ *and* $0 \leq r \leq p-1$. *Then*

$$\sum_{b\in B} b^r - \sum_{c\in C} c^r = \sum_{k=0}^{2^{p+1}-1} a_{k+1}(a+kd)^r = \sum_{k=0}^{4^{m+1}-1} a_{k+1}(a+kd)^r = T_{m,r}(a)$$

If $r = 2r_1$, *then* $2r_1 \leq 2m \Longrightarrow r_1 \leq m$ *and consequently* $T_{m,r}(z) = T_{m,2r_1}(z) = 0$. *When* $r = 2r_1+1$, *then* $2r_1+1 \leq 2m \Longrightarrow r_1 < m$ *and again* $T_{m,r}(z) = T_{m,2r_1+1}(z) = 0$.
The second case $p = 2m$ *and* $0 \leq r \leq 2m-1$. *Then*

$$\sum_{b\in B} b^r - \sum_{c\in C} c^r = \sum_{k=0}^{2^{2m+1}-1} a_{k+1}(a+kd)^r = \sum_{k=0}^{4^m-1} a_{k+1}(a+kd)^r + \sum_{k=4^m}^{2.4^m-1} a_{k+1}(a+kd)^r =$$

$$= T_{m-1,r}(a) + \sum_{k=0}^{4^m-1} a_{2^{2m}+k+1}((a+4^m d)+kd)^r = T_{m-1,r}(a) - T_{m-1,r}(a+4^m d)$$

If $r = 2r_1$, *then* $2r_1 \leq 2m-1 \Longrightarrow r_1 \leq m-1$ *and therefore* $T_{m-1,r}(z) = T_{m-1,2r_1}(z) = 0$.
If $r = 2r_1+1$, *then* $2r_1+1 \leq 2m-1 \Longrightarrow r_1 \leq m-1$ *and again* $T_{m-1,r}(z) = T_{m-1,2r_1+1}(z) = 0$, *thus corollary 1 is proved.* □

3.2 Proof of proposition 1

It is clear from the proof of 1 that the constants a and d can be replaced by elements of an arbitrary field, having characteristic 0. We will consider the field of rational functions with complex coefficients, replacing a and d with rational functions $P(z)$ and $Q(z)$. Set $R_k(z) = P(z) + kQ(z)$ and when $n = 2s,\ m = 0, 1, \ldots, 2s-1$, then simple calculation gives

$$H_{n,m}(z) = H_{2s,m}(z) = \sum_{l=2s}^{\infty} \sum_{k=1}^{2^l} a_k \left(P(z) + k.Q(z)\right)^m = \sum_{l=2s}^{\infty} \sum_{k=0}^{2^l-1} a_{k+1} R_{k+1}^m(z) =$$

$$= \sum_{l=s}^{\infty} \left(\sum_{k=0}^{2^{2l}-1} a_{k+1} R_{k+1}^m(z) + \sum_{k=0}^{2^{2l+1}-1} a_{k+1} R_{k+1}^m(z) \right) =$$

$$= \sum_{l=s}^{\infty} \left(T_{l-1,m}(R_1(z)) + \sum_{k=0}^{2^{2l}-1} a_{k+1} R_{k+1}^m(z) + \sum_{k=2^{2l}}^{2^{2l+1}-1} a_{k+1} R_{k+1}^m(z) \right) =$$

$$= \sum_{l=s}^{\infty} \left(2T_{l-1,m}(R_1(z)) + \sum_{k=0}^{2^{2l}-1} a_{2^{2l}+k+1} R^m_{2^{2l}+k+1}(z) \right) =$$

$$= \sum_{l=s}^{\infty} (2T_{l-1,m}(R_1(z)) - T_{l-1,m}(R_{2^{2l}+1}(z))) = 0,$$

$$\text{since } l-1 \geq s-1 = \left[\frac{2s-1}{2}\right] \geq \left[\frac{m}{2}\right]$$

In the case $n = 2s+1,\ m = 0, 1, \ldots, 2s$, we have

$$H_{n,m}(z) = H_{2s+1,m}(z) = \sum_{l=2s+1}^{\infty} \sum_{k=0}^{2^l-1} a_{k+1} R^m_{k+1}(z) =$$

$$= \sum_{l=s}^{\infty} \left(\sum_{k=0}^{2^{2(l+1)}-1} a_{k+1} R^m_{k+1}(z) + \sum_{k=0}^{2^{2l+1}-1} a_{k+1} R^m_{k+1}(z) \right) =$$

$$= \sum_{l=s}^{\infty} \left(T_{l,m}(R_1(z)) + \sum_{k=0}^{2^{2l}-1} a_{k+1} R^m_{k+1}(z) + \sum_{k=2^{2l}}^{2^{2l+1}-1} a_{k+1} R^m_{k+1}(z) \right) =$$

$$= \sum_{l=s}^{\infty} \left(2T_{l,m}(R_1(z)) + \sum_{k=0}^{2^{2l}-1} a_{2^{2l}+k+1} R^m_{2^{2l}+k+1}(z) \right) =$$

$$= \sum_{l=s}^{\infty} (2T_{l,m}(R_1(z)) - T_{l,m}(R_{2^{2l}+1}(z))) = 0,$$

$$\text{since } l \geq s \geq \left[\frac{m}{2}\right]$$

Now set $n = m = 2s$. According to calculations above and proposition 6, we have

$$H_{n,m}(z) = H_{2s,2s}(z) = \sum_{l=s}^{\infty} (2T_{l-1,2s}(R_1(z)) - T_{l-1,2s}(R_{2^{2l}+1}(z))) =$$

$$= 2T_{s-1,2s}(R_1(z)) - T_{s-1,2s}(R_{2^{2s}+1}(z)) = (2s)! 2^{2s^2-s} Q^{2s}(z) = n! 2^{\frac{n^2-n}{2}} Q^n(z),$$

thus 1 is proved. □

3.3 Proof of proposition 2

Let $n = m = 2s$ and set $\gamma_n = n!2^{\frac{n^2-n}{2}}$. According to proposition 1 we have

$$H_{n,n}(z) = \gamma_n Q^n(z) \Longrightarrow \frac{d}{dz} H_{n,n}(z) = n\gamma_n Q^{n-1}(z)Q'(z) \Longrightarrow$$

$$\Longrightarrow \gamma_n Q^{n-1}(z)Q'(z) = \sum_{l=2s}^{\infty}\sum_{k=1}^{2^l} a_k R_k^{n-1}(z)R_k'(z) = \sum_{l=s}^{\infty}\sum_{k=1}^{2^{2l}} a_k R_k^{n-1}(z)R_k'(z) =$$

$$= \sum_{k=1}^{2^{2s}} a_k R_k^{n-1}(z)R_k'(z).$$

and by induction, we prove that

$$\gamma_n Q(z)Q'(z)Q''(z)\dots Q^{(n-1)}(z) = \sum_{k=1}^{2^{2s}} a_k R_k(z)R_k'(z)R_k''(z)\dots R_k^{(n-1)}(z) \Longrightarrow$$

$$\Longrightarrow \gamma_n = \sum_{k=1}^{2^{2s}} a_k \left(\frac{P(z)}{Q(z)} + k\right)\left(\frac{P'(z)}{Q'(z)} + k\right)\left(\frac{P''(z)}{Q''(z)} + k\right)\dots\left(\frac{P^{(n-1)}(z)}{Q^{(n-1)}(z)} + k\right)$$

The last equality holds for all polynomials P and those Q for which $\deg Q \geq n-1$. For arbitrary $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$, we can choose P and Q, such that for fixed z_0 to satisfy the equalities:

$$P(z_0) - \alpha_1 Q(z_0) = P'(z_0) - \alpha_2 Q'(z_0) = P''(z_0) - \alpha_3 Q''(z_0) = \dots =$$

$$= P^{(n-1)}(z_0) - \alpha_n Q^{(n-1)}(z_0) = 0,$$

where $Q^{(i)}(z_0) \neq 0,\ i = 0, 1, \dots, n-1$. Therefore

$$\gamma_n = \sum_{k=1}^{2^n} a_k(\alpha_1 + k)(\alpha_2 + k)\cdots(\alpha_n + k).$$

Remark 1. For even n and arbitrary rational functions $S_1(z), S_2(z), \dots, S_n(z)$ is valid:

$$\gamma_n = \sum_{k=1}^{2^n} a_k(S_1(z) + k)(S_2(z) + k)\cdots(S_n(z) + k),$$

which is stronger than proved above.

Corollary 2 *For every polynomial $P \in \mathbb{C}[z]$ from even degree $n = \deg P$ and leading coefficient α is valid*

$$\sum_{k=1}^{2^{\deg P}} a_k P(k) = \alpha.n!2^{\frac{n^2-n}{2}}$$

3.4 Proof of proposition 3

We can describe a proof based on the results obtained above for the sequence $\{H_{n,m}(z)\}$, but we prefer another approach. Let us introduce the polynomial sequence $\{\widetilde{T}_{s,p}(z_1,\ldots,z_p)\}$ by:

$$\widetilde{T}_{s,p}(z_i)=\widetilde{T}_{s,p}(z_1,z_2,\ldots,z_p)=\sum_{k=0}^{4^{s+1}-1}a_{k+1}\prod_{i=1}^{p}(z_i+kd),$$

which is generalization of $\{T_{s,p}(z)\}$. The following recurrent formula is derived similarly as above:

$$\widetilde{T}_{s,p}(z_i)=\widetilde{T}_{s-1,p}(z_i+3.4^s d)-\widetilde{T}_{s-1,p}(z_i+2.4^s d)-\widetilde{T}_{s-1,p}(z_i+4^s d)+\widetilde{T}_{s-1,p}(z_i)$$

We will prove that $\widetilde{T}_{k,p}(z_1,z_2,\ldots,z_p)\equiv 0$, for $k\geq\left[\frac{p}{2}\right]$, while in the case $1\leq k\leq\left[\frac{p}{2}\right]-1$ is valid

$$\widetilde{T}_{k,p}(z_1,z_2,\ldots,z_p)=$$

$$=\sum_{i_1=2k}^{p-2}\sum_{i_2=2(k-1)}^{i_1-2}\cdots\sum_{i_{k+1}=0}^{i_k-2}\sum_{\sigma(k+1)\subset\sigma(k)\subset\cdots\subset\sigma(1)}L_{k,p}d^{p-i_{k+1}}z_{\sigma_1(k+1)}\ldots z_{\sigma_{i_{k+1}}(k+1)}$$

and $L_{k,p}$ is defined as above, and set $z_{\sigma_0(k+1)}$ equal to 1.
Assuming that $z_{i_0}=1$ and calculate

$$\widetilde{T}_{0,p}(z_j)=z_1z_2\ldots z_p+\sum_{s=0}^{p}\sum_{0\leq j_1<\cdots<j_s\leq p}(3^{p-s}-2^{p-s}-1)d^{p-s}z_{j_1}z_{j_2}\ldots z_{j_s}=$$

$$=\sum_{s=0}^{p-2}\sum_{0\leq j_1<\cdots<j_s\leq p}(3^{p-s}-2^{p-s}-1)d^{p-s}z_{j_1}z_{j_2}\ldots z_{j_s}=$$

$$=\sum_{i_1=0}^{p-2}\sum_{\sigma(1)}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}z_{\sigma_1(1)}z_{\sigma_2(1)}\ldots z_{\sigma_{i_1}(1)}.$$

In the case $k=1$ we get

$$\widetilde{T}_{1,p}(z_j)=\sum_{i_1=2}^{p-2}\sum_{i_2=0}^{i_1-2}\sum_{\sigma(2)\subset\sigma(1)}L_{1,p}d^{p-i_2}z_{\sigma_1(2)}z_{\sigma_2(2)}\ldots z_{\sigma_{i_2}(2)},$$

whereby the statement is established for $k=1$. Assume that, for some $k\geq 2$, $\widetilde{T}_{k-1,p}(z_j)$ satisfies the recurrent formula and denote

$$G^{l,p}_{j_1,j_2,\ldots,j_{l+1}}=d^{p-i_{l+1}}L_{l,p}\text{ when }l\geq 1.$$

Then

$$\widetilde{T}_{k,p}(z_i) = \widetilde{T}_{k-1,p}(z_i+3.4^k d) - \widetilde{T}_{k-1,p}(z_i+2.4^k d) - \widetilde{T}_{k-1,p}(z_i+4^k d) + \widetilde{T}_{k-1,p}(z_i) =$$

$$= \sum_{i_1=2(k-1)}^{p-2} \sum_{i_2=2(k-2)}^{i_1-2} \cdots \sum_{i_k=0}^{i_{k-1}-2} \sum_{\sigma(k)\subset\sigma(k-1)\subset\cdots\subset\sigma(1)} L_{k,p} d^{p-i_k} \Psi(z_{\sigma_1(k)} z_{\sigma_2(k)} \dots z_{\sigma_{i_k}(k)})$$

where

$$\Psi(z_{\sigma_1(k)} z_{\sigma_2(k)} \dots z_{\sigma_{i_k}(k)}) = \prod_{i=1}^{i_k}(z_{\sigma_i(k)}+3.4^k d) - \prod_{i=1}^{i_k}(z_{\sigma_i(k)}+2.4^k d) - \prod_{i=1}^{i_k}(z_{\sigma_i(k)}+4^k d) +$$

$$+ z_{\sigma_1(k)} z_{\sigma_2(k)} \dots z_{\sigma_{i_k}(k)} = z_{\sigma_1(k)} z_{\sigma_2(k)} \dots z_{\sigma_{i_k}(k)} +$$

$$+ \sum_{t=0}^{i_k} \sum_{j_1<\cdots<j_t} (3^{i_k-t} - 2^{i_k-t} - 1) 4^{k(i_k-t)} d^{i_k-t} z_{j_1} \dots z_{j_t},$$

here $0 \le j_1 < j_2 < \cdots < j_t \le p$ and $\{j_1, j_2, \dots, j_t\}$ runs over all t-element subsets of $\{\sigma_1(k), \sigma_2(k), \dots, \sigma_{i_k}(k)\}$, for all $t = 1, 2, \dots, i_k$. We write $\sigma(k+1) \subset \sigma(k)$ to denote the aforementioned inclusion of sets, and put $i_{k+1} = t$. For every l let $\sigma(l)$ be the i_l-member subsets of $\{1, 2, \dots, p\}$. Therefore

$$\Psi(z_{\sigma_1(k)} \dots z_{\sigma_{i_k}(k)}) =$$

$$\sum_{i_{k+1}=0}^{i_k-2} \sum_{\sigma(k+1)\subset\sigma(k)} (3^{i_k-i_{k+1}} - 2^{i_k-i_{k+1}} - 1) 4^{k(i_k-i_{k+1})} d^{i_k-i_{k+1}} z_{\sigma_1(k+1)} \dots z_{\sigma_{i_{k+1}}(k+1)}$$

whence it immediately follows that $\widetilde{T}_{k,p}(z_1, z_2, \dots, z_p)$ has the desired form. Let us determine the degree of $\widetilde{T}_{k,p}(z),\ k \ge 0$. According to the formula above, one has $i_{k+1} \le i_k - 2 \le i_{k-1} - 4 \le \cdots \le i_1 - 2k \le p - 2(k+1)$, as equality holds everywhere. Thus $\deg \widetilde{T}_{k,p}(z) = p - 2(k+1)$.
Let $p = 2m + r,\ r \in \{0, 1\}$, then

$$\deg \widetilde{T}_{m-1,p}(z) = p - 2m = r.$$

In the case $r = 0$ we obtain that $\widetilde{T}_{m-1,p}(z_1, z_2 \dots, z_p)$ is a constant:

$$\widetilde{T}_{m-1,p}(z_1, z_2 \dots, z_p) = \widetilde{T}_{m-1,p}(0, 0, \dots, 0) = \sum_{j=0}^{4^m-1} a_{j+1} \prod_{i=1}^{p} jd = \sum_{j=0}^{4^m-1} a_{j+1} j^p d^p =$$

$$= T_{m-1,p}(0) = p! 2^{\frac{p^2-p}{2}} d^p, \quad p = 2m.$$

When $r = 1$, analogous to the proof for $T_{m,2m+1}(z) = 0$, we can prove

$$\widetilde{T}_{m,2m+1}(z_1, z_2 \dots, z_{2m+1}) \equiv 0.$$

Therefore

$$\widetilde{T}_{m,2m}(z_1, z_2 \ldots, z_{2m}) = \widetilde{T}_{m,2m+1}(z_1, z_2 \ldots, z_{2m+1}) = 0.$$

Using the recurrent formula again

$$\widetilde{T}_{k,p}(z_i) = \widetilde{T}_{k-1,p}(z_i + 3.4^k d) - \widetilde{T}_{k-1,p}(z_i + 2.4^k d) - \widetilde{T}_{k-1,p}(z_i + 4^k d) + \widetilde{T}_{k-1,p}(z_i)$$

we obtain $\widetilde{T}_{k,p}(z_1, z_2 \ldots, z_p) \equiv 0$ for $k \geq \left[\frac{p}{2}\right]$.

From the results above, the proposition 3 easily follows. Indeed, let $\deg P = n$, $P(z) = \alpha(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$ and set $d = 1$ in the definition of $\widetilde{T}_{k,p}$. In the case $n = 2m+1$:

$$\sum_{k=1}^{2^{1+\deg P}} a_k P(k) = \sum_{k=1}^{2^{2m+2}} a_k P(k) = \sum_{k=0}^{2^{2m+2}-1} a_{k+1} P(k+1) =$$

$$= \alpha. \sum_{k=0}^{2^{2m+2}-1} a_{k+1} \prod_{i=1}^{2m+1} (1 - \alpha_i + k) =$$

$$= \alpha.\widetilde{T}_{m,2m+1}(1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_{2m+1}) = 0.$$

In the case $n = 2m$ analogously

$$\sum_{k=1}^{2^{1+\deg P}} a_k P(k) = \sum_{k=0}^{2^{2m+1}-1} a_{k+1} P(k+1) = \sum_{k=0}^{4^m-1} a_{k+1} P(k+1) + \sum_{k=4^m}^{2.4^m-1} a_{k+1} P(k+1) =$$

$$= \alpha.\widetilde{T}_{m-1,2m}(1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_{2m}) + \sum_{k=0}^{4^m-1} a_{4^m+k+1} P(4^m + k + 1) =$$

$$= \alpha.n!2^{\frac{n^2-n}{2}} - \sum_{k=0}^{4^m-1} a_{k+1} P(4^m + k + 1) =$$

$$= \alpha.n!2^{\frac{n^2-n}{2}} - \alpha.\widetilde{T}_{m-1,2m}(4^m + 1 - \alpha_1, 4^m + 1 - \alpha_2, \ldots, 4^m + 1 - \alpha_{2m}) = 0,$$

which completes the proof. The calculations above show that

$$\sum_{k=1}^{2^{\deg P}} a_k P(k) = \alpha.\widetilde{T}_{m-1,2m}(1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_{2m}) = \alpha.n!2^{\frac{n^2-n}{2}},$$

when $\deg P = n = 2m$,

which proves the proposition 2. Therefore for arbitrary complex variables

$$z_1, z_2, \ldots, z_{2m},$$

we will have

$$\sum_{k=1}^{2^n} a_k (z_1 + k)(z_2 + k) \cdots (z_n + k) = n!2^{\frac{n^2-n}{2}}.$$

□

3.5 Proof of proposition 4

Let $l = \lceil \log_2 k \rceil$. We will use the following result: for every fixed $m \in \mathbb{N}$, there is a m-member arithmetic progression of prime numbers. Let $q_1, q_2, \dots, q_{l.2^{p+1}}$ to be $l.2^{p+1}-$member arithmetic progression of prime numbers and let us define the sets:

$$A_i = \{q_{(i-1)2^{p+1}+1}, q_{(i-1)2^{p+1}+2}, \dots, q_{i.2^{p+1}},\}, \text{ for } i = 1, 2, \dots, l.$$

Applying corollary 1 to each of the sets A_i:

$$\exists B_i, C_i \subset A_i : \; B_i \cap C_i = \emptyset, \; B_i \cup C_i = A_i, \; |B_i| = |C_i| = \frac{|A_i|}{2} = 2^p \text{ and}$$

$$(*) \; \sum_{q \in B_i} q^r = \sum_{q \in C_i} q^r, \; r = 0, 1, \dots, p; \; i = 1, 2, \dots, l.$$

Define the number

$$N(p,k,r) = \prod_{i=1}^{l} \left(\sum_{q \in B_i} q^r \right) = \left(\sum_{q \in B_1} q^r \right) \left(\sum_{q \in B_2} q^r \right) \cdots \left(\sum_{q \in B_l} q^r \right)$$

It is straightforward that $N(p,k,r)$ is equal to sum of 2^{pl} numbers of the type $(q_{j_1} q_{j_2} \dots q_{j_l})^r$, i.e.

$$(**) \; N(p,k,r) = \sum_{j_1, j_2, \dots, j_l} (q_{j_1} q_{j_2} \dots q_{j_l})^r,$$

where the summation is taken on all $j_1, j_2, \dots, j_l$ with $q_{j_k} \in B_k, \; 1 \le k \le l, \; 1 \le j_k \le 2^p$.

According to $(*)$ the number $N(p,k,r)$ can be represented in the form $(**)$, where the summation is taken on all $j_1, j_2, \dots, j_l$, but $q_{j_k} \in B_k$ or $q_{j_k} \in C_k, \; 1 \le k \le l$. Consequently $N(p,k,r)$ is represented at least in 2^l different ways as sum of r-powers of different square-free positive integers (exactly 2^{pl} in number). On the other hand $2^l = 2^{\lceil \log_2 k \rceil} \ge 2^{\log_2 k} = k$ and $n = 2^{pl} = 2^{p \lceil \log_2 k \rceil}$. It is now clear how to dermine the numbers x_{ij} with the required property. Let $\sigma(i) = (\sigma_1(i), \sigma_2(i), \dots, \sigma_l(i)), \; i = 1, 2, \dots 2^l$ be all 2^l in number l-member sequences of 0 and 1. For all $i, k, j_k : \; 1 \le i \le 2^l, \; 1 \le k \le l, \; 1 \le j_k \le 2^p$ we define $q_{j_k, \sigma_k(i)} \in B_k$ when $\sigma_k(i) = 0$, and $q_{j_k, \sigma_k(i)} \in C_k$ when $\sigma_k(i) = 1$. Therefore

$$N(p,k,r) = \sum_{j_1, j_2, \dots, j_l} \left(q_{j_1, \sigma_1(1)} q_{j_2, \sigma_2(1)} \dots q_{j_l, \sigma_l(1)} \right)^r =$$

$$= \sum_{j_1, j_2, \dots, j_l} \left(q_{j_1, \sigma_1(2)} q_{j_2, \sigma_2(2)} \dots q_{j_l, \sigma_l(2)} \right)^r =$$

$$= \dots = \sum_{j_1, j_2, \dots, j_l} \left(q_{j_1, \sigma_1(2^l)} q_{j_2, \sigma_2(2^l)} \dots q_{j_l, \sigma_l(2^l)} \right)^r, \; \forall r = 0, 1, \dots p.$$

For each fixed i, leaving k and j_k to run over $1, 2, \ldots, l$ and $1, 2, \ldots, 2^p$ respectively, the numbers $q_{j_1,\sigma_1(i)} q_{j_2,\sigma_2(i)} \ldots q_{j_l,\sigma_l(i)}$ are $n = 2^{pl}$ in number and we can rearrange them in ascending order (because there are two by two distinct). For fixed i, we define the number x_{ij} as j-th of the largest among the numbers $q_{j_1,\sigma_1(i)} q_{j_2,\sigma_2(i)} \ldots q_{j_l,\sigma_l(i)}$, for $j = 1, 2, \ldots, 2^{pl}$. Performing this procedure for $i = 1, i = 2, \ldots, i = 2^l$ we obtain numbers x_{ij} satisfying the equations in proposition 4. $\square$

4 Arithmetic progression theorem

4.1 Product of arithmetic progressions

Let $\{b_i\}_{i=0}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ be arbitrary sequences of complex numbers. We define the sets

$$\gamma_m = \{b_m + kd_m \mid k \in \mathbb{Z}\} \text{ and } \Gamma_n = \gamma_1 \gamma_2 \ldots \gamma_n = \left\{ \prod_{i=1}^{n} (b_i + kd_i) \mid k \in \mathbb{Z} \right\},$$

where n is a positive integer.

Theorem 1. *Each m-element subset $\Gamma_{m,n}$ of Γ_n, where $2^{n+1}|m$ and k runs through m consecutive integers, can be represented as disjoint union of two subsets, with equal sums of the elements.*

Proof 3 *Applying the proposition* 3 *for* $P(z) = (d_1 z + b_1)(d_2 z + b_2) \ldots (d_n z + b_n)$ *leads to*

$$0 = \sum_{k=1}^{2^{n+1}} a_k P(k) = \sum_{k=1}^{2^{n+1}} a_k \prod_{i=1}^{n} (b_i + kd_i),$$

which proves the statement $m = 2^{n+1}$. The general case $m = s.2^{n+1}$ follows directly.

We realize a second proof throughsequence of polynomials $\{t_{s,n}\}_{s,n \geq 1}$, defined by

$$t_{s,n}(z_1, z_2, \ldots, z_n) = \sum_{k=0}^{4^{s+1}-1} a_{k+1} \prod_{i=1}^{n} (z_i + kd_i).$$

The relation between the polynomials $t_{s,n}$ and $\widetilde{T}_{s,n}$ is given by

$$t_{s,n}(z_1, z_2, \ldots, z_n) = \frac{d_1 d_2 \ldots d_n}{d^n} \widetilde{T}_{s,n} \left(\frac{z_1 d}{d_1}, \frac{z_2 d}{d_2}, \ldots, \frac{z_n d}{d_n} \right),$$

from which it follows that

$$t_{s,n}(z_1, z_2, \ldots, z_n) \equiv 0 \text{ when } s \geq \left[\frac{n}{2} \right].$$

The statement follows easily, by considering two cases depending on the parity of n. $\square$

4.2 Generalization of arithmetic progression theorem

Theorem 2. *Let n and m be positive integers. Then there exists a positive integer $s = s(n, m)$ with the property: each s-element subset of Γ_n, where k runs through s consecutive integers, can be represented as disjoint union of m subsets, with equal sums of elements for any such subset.*

We will prove a special case of theorem 2, which is obtained for

$$\gamma_1 = \gamma_2 = \cdots = \gamma_n = \gamma \text{ and } \Gamma_n = \gamma^n = \{(a + kd)^n : \ k \in \mathbb{Z}\},$$

and any positive integer s, that is divisible by $2^{\left[\frac{n-1}{2}\right]+1} m^{\lceil \frac{n-1}{2} \rceil + 1}$ has the desired property. For $p \in \mathbb{N}$ we define the sequence of maps $\{\varphi_{n,m}(z)\}_{n,m \geq 1}$ by

$$\varphi_{n,m} : \mathbb{C} \longrightarrow \mathbb{C}^m$$

$$z \xrightarrow{\varphi_{0,m}} (A_{1,2}(z), A_{2,3}(z), \ldots, A_{m-1,m}(z), A_{1,m}(z)),$$

where for $l = 1, 2, \ldots, m-1$ we put

$$A_{l,l+1}(z) = (z + (l-1)d)^p - (z + l.d)^p - (z + (2m - l - 1)d)^p + (z + (2m - l)d)^p,$$

$$A_{1,m}(z) = \sum_{l=1}^{m-1} A_{l,l+1}(z).$$

We will prove that for the set $\Gamma_p = \gamma^p = \{(a + kd)^p : \ k \in \mathbb{Z}\}$, the number $s = 2^{\left[\frac{p-1}{2}\right]+1} m^{\lceil \frac{p-1}{2} \rceil + 1}$ has the desired property.

Let $e_1, e_2, \ldots, e_m$ be a basis for $\mathbb{C}^m$ and define $\tau \in \mathbf{Hom}(\mathbb{C}^m)$ by:

$$\tau(e_1) = e_2, \ \tau(e_2) = e_3, \ldots, \tau(e_{m-2}) = e_{m-1}; \ \tau(e_{m-1}) = -e_m, \ \tau(e_m) = -e_1.$$

A direct calculation shows that $\tau^m = \mathbf{id}_{\mathbb{C}^m}$ and put

$$\varphi_{2s,m}(z) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \left(\varphi_{2s-2,m}(z + 2^s m^s j.d) + \varphi_{2s-2,m}(z + 2^s m^s (2m - 1 - j)d)\right), \ s \geq 1;$$

$$\varphi_{2s+1,m}(z) = \sum_{j=0}^{m-1} \tau^j \circ \varphi_{2s,m}(z + 2^{s+1} m^{s+1} j.d), \ s \geq 0.$$

A straightforward calculation gives:

$$\varphi_{0,m}(z) = \sum_{l=1}^{m-1} A_{l,l+1}(z)(e_l + e_m) =$$

$$\sum_{l=1}^{m-1}\sum_{k=0}^{p} r(k,l)z^k(e_l+e_m) = \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)z^k(e_l+e_m)$$
$$\Longrightarrow \varphi_{0,m}(z) = \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)z^k(e_l+e_m),$$

where

$$r(k,l) = \binom{p}{k}\left((l-1)^{p-k} - l^{p-k} - (2m-l-1)^{p-k} + (2m-l)^{p-k}\right)d^{p-k}.$$

Since the coordinate functions of $\varphi_{n,m}$ are polynomials of z of the same degree, depending on n (i.e. $\deg A_{l,l+1} = \deg A_{1,m}$ for all l), then we can use the notion for the *degree* of the map $\varphi_{n,m}$, by setting $\deg \varphi_{n,m}$ to be equal to the the *degree* of its coordinate functions. We will prove that

$$\deg \varphi_{n,m} = p-n-2, \text{ for } n \le p-2,$$

$$\varphi_{n,m}(z) \equiv \mathbf{0}, \text{ for } n \ge p-1.$$

For $n=0$ according to calculations above we have $\deg \varphi_{0,m} = p-2$. For $n=1$ we put $Q_k(z,t) = (z+2m.d.t)^k$ and calculate

$$\varphi_{1,m}(z) = \sum_{j=0}^{m-1} \tau^j \circ \varphi_{0,m}(z+2m.j.d) = \sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)(z+2m.j.d)^k(e_l+e_m)$$

$$= \sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)(Q_k(z,j)-z^k)(e_l+e_m) + \sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)z^k(e_l+e_m)$$

$$\Longrightarrow \varphi_{1,m}(z) = \sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)(Q_k(z,j)-z^k)(e_l+e_m), \text{ since}$$

$$(*)\quad \sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)z^k(e_l+e_m) = \mathbf{0}.$$

Therefore $\deg \varphi_{1,m}(z) = \deg(Q_{p-2}(z,j) - z^{p-2}) = p-3$. We define

$$\psi = \sum_{j=0}^{m-1} \tau^j, \; K = \{a_1e_1 + a_2e_2 + \cdots + a_{m-1}e_{m-1} + \sum_{j=1}^{m-1} a_je_m : \; a_j \in \mathbb{C}\}.$$

We will prove that $K \subset \operatorname{Ker}\psi$, from which immediately follows the equality $(*)$. We calculate

$$\psi(e_1) = \psi(e_2) = \cdots = \psi(e_{m-1}) = -\psi(e_m) = -e_m + \sum_{j=1}^{m-1} e_j$$

$$\Rightarrow \psi(a_1e_1+a_2e_2+\cdots a_{m-1}e_{m-1}) = \sum_{j=1}^{m-1} a_j\psi(e_1) = -\sum_{j=1}^{m-1} a_j\psi(e_m) \Rightarrow K \subset \operatorname{Ker}\psi.$$

For $n = 2$ we obtain:

$$\varphi_{2,m}(z) = \sum_{j=0}^{m-1} \tau^j \circ (\varphi_{0,m}(z+2mj.d) + \varphi_{0,m}(z+2m(2m-1-j)d)) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)(Q_k(z,j) + Q_k(z,2m-1-j))(e_l+e_m) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)(Q_k(z,j)+Q_k(z,2m-1-j)-2z^k-2kdm(2m-1)z^{k-1})(e_l+e_m),$$

$$\text{since } \sum_{j=0}^{m-1} \tau^j \circ \sum_{l=1}^{m-1}\sum_{k=0}^{p-2} r(k,l)(2z^k+2kdm(2m-1)z^{k-1})(e_l+e_m) = \mathbf{0}.$$

Therefore

$$\deg\varphi_{2,m}(z) = \deg(Q_{p-2}(z,j)+Q_{p-2}(z,2m-1-j)-2z^{p-2}-2kdm(2m-1)z^{p-3}) =$$
$$p-4.$$

Let $\deg\varphi_{2s,m}(z) = p-2-2s$ be valid for some $s < \left[\frac{p-2}{2}\right]$. Then $\varphi_{2s,m}$ has the form

$$\varphi_{2s,m}(z) = \sum_{k=0}^{p-2-2s}\sum_{i=1}^{m-1} \alpha_{k,i} z^{p-2-2s-k}\beta_i(e_i+e_m),\ \ \alpha_0, \beta_i \neq 0.$$

We calculate:

$$\varphi_{2s+1,m}(z) = \sum_{j=0}^{m-1} \tau^j \circ \varphi_{2s,m}(z+2^{s+1}m^{s+1}j.d) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{k=0}^{p-2-2s}\sum_{i=1}^{m-1} \alpha_{k,i}(z+2^{s+1}m^{s+1}j.d)^{p-2-2s-k}\beta_i(e_i+e_m) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{k=0}^{p-2-2s}\sum_{i=1}^{m-1} \left(\alpha_{k,i}(z+2^{s+1}m^{s+1}j.d)^{p-2-2s-k} - \alpha_{k,i}z^{p-2-2s-k}\right)\beta_i(e_i+e_m),$$

where the last equality is due to

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{k=0}^{p-2-2s}\sum_{i=1}^{m-1} \alpha_{k,i} z^{p-2-2s-k}\beta_i(e_i+e_m) = \mathbf{0}.$$

Therefore

$$\deg\varphi_{2s+1,m} = \deg(\alpha_{0,i}(z+2^{s+1}m^{s+1}j.d)^{p-2-2s} - \alpha_{0,i}z^{p-2-2s}) = p-3-2s.$$

We put $t = p - 2 - 2s$ and calculate

$$\varphi_{2s+2,m}(z) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \left(\varphi_{2s,m}(z + 2^{s+1}m^{s+1}j.d) + \varphi_{2s,m}(z + 2^{s+1}m^{s+1}(2m-1-j)d)\right) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{k=0}^{t} \sum_{i=1}^{m-1} \alpha_{k,i} \left((z + 2^{s+1}m^{s+1}j.d)^{t-k} + (z + 2^{s+1}m^{s+1}(2m-1-j)d)^{t-k}\right) \beta_i(e_i + e_m) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{k=0}^{t} \sum_{i=1}^{m-1} \alpha_{k,i} \left(Q_{t-k}(z, 2^s m^s j) + Q_{t-k}(z, 2^s m^s(2m-1-j))\right) \beta_i(e_i + e_m) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{k=0}^{t} \sum_{i=1}^{m-1} \alpha_{k,i} \left(R_{t-k}(z, j) - 2z^{t-k} - 2^s m^s d(2m-1)(t-k)z^{t-k-1}\right) \beta_i(e_i + e_m),$$

where $R_{t-k}(z, j) = Q_{t-k}(z, 2^s m^s j) + Q_{t-k}(z, 2^s m^s(2m-1-j))$.

Finally

$$\deg \varphi_{2s+2,m} = \deg \left(R_t(z, j) - 2z^t - 2^s m^s t d(2m-1)z^{t-1}\right) = t - 2 = p - 4 - 2s.$$

Consequently $\deg \varphi_{p-2,m} = 0$ and

$$\varphi_{p-2,m}(z) = \sum_{i=1}^{m-1} \beta_i(e_i + e_m) \text{ and } \varphi_{p-3,m}(z) = z \sum_{i=1}^{m-1} \gamma_i(e_i + e_m) \Longrightarrow$$

$$\varphi_{p-1,m}(z) =$$

$$= \sum_{j=0}^{m-1} \tau^j \circ \Big(\varphi_{p-3,m}(z + 2^{\frac{p-1}{2}} m^{\frac{p-1}{2}} j.d) + \varphi_{p-3,m}(z + 2^{\frac{p-1}{2}} m^{\frac{p-1}{2}}(2m-1-j)d)\Big) =$$

$$\sum_{j=0}^{m-1} \tau^j \circ \sum_{i=1}^{m-1} \Big(2z + 2^{\frac{p-1}{2}} m^{\frac{p-1}{2}}(2m-1)d\Big) \sum_{j=0}^{m-1} \gamma_i(e_i + e_m) = \mathbf{0}, \text{ when } p - 1 = 2s.$$

Let us consider the case $p - 1 = 2s + 1$:

$$\varphi_{p-1,m}(z) = \sum_{j=0}^{m-1} \tau^j \circ \varphi_{p-2,m}(z + 2^{\frac{p}{2}} m^{\frac{p}{2}} j.d) = \sum_{j=0}^{m-1} \tau^j \circ \sum_{i=1}^{m-1} \beta_i(e_i + e_m) = \mathbf{0},$$

which completes the proof of theorem 2 in the considered special case. □

Corollary 3 *Let m and n be positive integers, $\Gamma = \{a + kd \mid k \in \mathbb{Z}\}$ is an arithmetic progression of complex numbers. Then any subset of $\Gamma' \subset \Gamma$ consisting of $2^{\left[\frac{n+1}{2}\right]} m^{\lceil \frac{n-1}{2} \rceil + 1}$ consecutive elements of Γ can be divided to m subsets, with equal sums of $r-$powers of elements (for any such subset), for $r = 1, 2, ..., n$.*

The proof follows directly from $\varphi_{n,m}(z) \equiv \mathbf{0}$, when $n \geq p - 1$, as the last result is derived in the proof of theorem 2.

5 Conclusion

Acknowledgment

Miroslav Stoenchev is supported by Bulgarian National Science Fund under Project KP-06-M32/2 - 17.12.2019 "Advanced Stochastic and Deterministic Approaches for Large-Scale Problems of Computational Mathematics". Venelin Todorov is supported by the Bulgarian National Science Fund under Projects KP-06-N52/5 "Efficient methods for modeling, optimization and decision making" and by the Bilateral Project KP-06-Russia/17 "New Highly Efficient Stochastic Simulation Methods and Applications".

References

1. J. -P. Allouche, J. O. Shallit, The ubiquitous Prouhet-Thue-Morse sequence. In C. Ding, T. Helleseth, and H. Niederreiter, editors, Sequences and Their Applications, Proceedings of SETA '98, pp. 1–16, Springer-Verlag, 1999.

2. P. Borwein, C. Ingalls, The Prouhet-Tarry-Escott Problem revisited. Enseign. Math. 40 (1994), 3–27. MR 95d:11038

3. P. Borwein, Computational Excursions in Analysis and Number Theory, Springer-Verlag, New York, 2002.

4. J. Byszewski and M. Ulas, Some identitites involving the Prouhet-Thue-Morse sequence and its relatives, Acta Math. Hungar., 127-2,438-456, 2015

5. H.L. Dorwart and O.E. Brown, The Tarry-Escott Problem, M.A.A. Monthly, 44, 613-626, 1937

6. G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth ed., Oxford Univ. Press, New York,1979

7. M. Stoenchev, V. Todorov, Combinatorial etude, Proceedings of the 16^{th} Conference on Computer Science and Intelligence Systems, pp. 231-234

8. T. Wakhare, C. Vignat, Settling some sum suppositions, arXiv:1805.10569 [math.NT]