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# System Gramians Evaluation for Hyperbolic PDEs

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**Abstract:** The paper considers the problem of gramians computation for linear hyperbolic distributed parameter systems. Two different cases are considered: vibrating string and beam systems. The presented approach is based on directly deriving the equations solutions by using time - space separation of variables and the Fourier series representation method. The initial problem framework is based on the state space formulation for infinite dimensional systems. This framework uses Riesz-spectral operators defined over Hilbert spaces and implements the concept of a $C_0$ strongly continuous semigroup generated by bounded system operator. The solution of the hyperbolic partial differential equations is divided in two parts. The zero input part is due to the initial conditions and participates in obtaining the observability gramian of the system. The zero state part is a consequence of the input signal effect and is used to compute the controllability gramian.



*Keywords:* distributed parameter system, hyperbolic partial differential equation, time-space separation, method of Fourier, vibrating string, vibrating beam, controllability and observability gramians

## 1. INTRODUCTION

In the system modeling and control literature, there exists a variety of methods for utilizing the gramians in the procedures of model order reduction, see for example Antoulas (2005). Although balanced model reduction has gained quite popularity for finite dimensional systems, it is still not the prevailing approach for model reduction of infinite dimensional systems. One of the main reasons for this drawback is the lack of efficient procedures for computing the gramians, especially for distributed parameter systems.

Distributed parameter systems are infinite dimensional systems and their solution belongs to the infinite dimensional Hilbert space. Most physical processes depend on both time and space variables and therefore, they are modeled as distributed parameter systems. Actually, the term distributed parameter follows from the condition that the corresponding solution reflects the distribution in space of physical quantities. A specific feature of such systems is their description in terms of irrational transfer functions, see Curtain and Morris (2009). A general framework for gramians definition for such systems is presented in Curtain and Zwart (2020), Glover et al. (1988). The system theory for infinite dimensional systems in the book of Curtain and Zwart (2020), is based on the concept of $C_0$-continuous semigroups for obtaining the solutions of the corresponding differential equations.

A model reduction procedure for expeditive and accurate solution of hyperbolic partial differential equations is presented in Taddei and Zhang (2021). A synthesized review of the time-space separation principle in modeling distributed parameter systems is given in Li and Qi (2010). The balanced truncation method for model order reduction of the semi discretized Stokes equation is presented in Stykel (2006). The undertaken approach is to obtain spatial discretization of the Stokes equation by using finite differences or the finite element method. The relationship between input/output and internal stability and the concepts of stabilizability/detectability are extended from the finite dimensional to the infinite dimensional case in Jacobson and Nett (2008). The generalization of the finite dimensional theory is made possible from the evolution of the state which is governed by strongly continuous semigroup of bounded linear operators on Hilbert spaces. The problem of model reduction for semistable infinite dimensional control systems is considered in Ziemann and Zhou (2019). Empirical gramians for distributed parameters systems are proposed in Jiang et al. (2018). Exact derivation of the observability gramian of an advection-diffusion PDE is investigated in Georges (2017). The state observers are designed for estimation and prediction of the pollution dynamics of a certain region by mainly improving sensor output sensitivity with respect to the initial state distribution. Another application of system gramians for optimal sensor and actuator placement is presented in Summers and Lygeros (2014). The authors have shown that a possible placement selection can be accomplished by optimization of certain controllability and observability metrics for a given network.

This paper considers the problem of gramians computation for vibrating string hyperbolic distributed parameter system. The solution is obtained directly by using the time-space separation principle and the Fourier series method. Both gramians are obtained in the general framework of $C_0$-continuous semigroup theory. The proposed technique


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for deriving the gramians is extended to the vibrating beam system.

## 2. MATHEMATICAL PRELIMINARIES ON HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS

Most models of mathematical physics are described by partial differential equations of second order. These models have been used for solving problems related to physical processes in fluid mechanics and electrodynamics. The general form of a second order partial differential equation is given as follows:

$$au_{tt} + bu_{tx} + cu_{xx} + du_t + eu_x + gu = f(x,t) \tag{1}$$

where $u = u(x,t)$ is the physical process under consideration and $f = f(x,t)$ is the external force applied to the process. The equation (1) is homogeneous if the external force on the right hand side is zero. If the condition $b^2 - 4ac > 0$ is satisfied, the equation (1) is called hyperbolic differential equation and the system describing the physical process governed by this equation is called hyperbolic distributed parameter system. The goal of this paper is to obtain explicit expressions for the gramians of certain hyperbolic distributed parameter systems. The abstract state equation of an arbitrary second order infinite dimensional system is given in the following form, see Curtain and Zwart (2020):

$$\dot{z}(t) = Az(t) + Bv(t), \tag{2}$$

where $z(t) = \left[ u(\cdot,t)\ \dfrac{du(\cdot,t)}{dt} \right]^T \in H$, $v(t) \in U$, $H$ is the infinite dimensional Hilbert space, $A$ is an operator acting on $H$, i.e. $A : H \to H$ and $B : U \to H$, where $U$ is the input space, which can be finite dimensional. The output equation of this system is presented as follows:

$$y(t) = Cz(t), \tag{3}$$

where $C : H \to Y$ with $Y$ being the output space, which also can be finite dimensional. The key concept for obtaining the solution of the infinite dimensional state equation, that generalizes the concept of state transition matrix, is the concept of a strongly continuous $C_0$ semigroup, see Curtain and Zwart (2020). A strongly continuous $C_0$ semigroup is a family of linear operators $S : H \to H$ which have the following properties: *i)* $S(0) = I$, where $I$ is the identity operator; *ii)* $S(t)S(s) = S(t+s)$; *iii)* $\lim_{t\to 0^+} S(t)z = z$ for every $z \in H$. The infinitesimal generator $A$ of a $C_0$ semigroup on $H$ is defined by the expression $Az = \lim_{t\to 0^+} \frac{1}{t}[S(t)z - z]$ with domain $D(A)$, that is the set of elements $z \in H$ for which the limit exists. In the Riesz-spectral operator framework, the operator $A$ can be presented in the form:

$$Az = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n, \tag{4}$$

where $\{\lambda_n, n = 1, 2, \ldots\}$ are the operator eigenvalues and $\{\phi_n, n = 1, 2, \ldots\}$ are the corresponding eigenfunctions, forming an orthonormal set of functions. The solution of the homogeneous equation is presented in terms of the $C_0$ semigroup $S(t)$ as follows:

$$z(t) = S(t)z_0, \tag{5}$$

where $z_0$ is the initial condition. In the Riesz-spectral operator framework, the solution of the homogeneous equation can be presented in the form:

$$z(t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle z_0, \phi_n \rangle \phi_n, \tag{6}$$

Similarly to the finite dimensional case, the controllability map on a finite interval $[0,t]$ is a bounded linear map $\mathcal{C} : H([0,t], U) \to H$, defined as:

$$(\mathcal{C}v)(t) = \int_0^t S(t-\tau)Bv(\tau)d\tau \tag{7}$$

The controllability gramian on the finite interval $[0,t]$ is given in operator form as $W_c(0,t) = \mathcal{C}\mathcal{C}^*$, where $\mathcal{C}^*$ is the adjoint operator of $\mathcal{C}$. Using expression (7), we can compute the controllability gramian on the time interval $[0,t]$ as an element of the space of linear maps $\mathcal{L}(H,H)$ from the expression, see Curtain and Zwart (2020):

$$W_c(0,t)z = \int_0^t S(\tau)BB^*S^*(\tau)zd\tau. \tag{8}$$

The infinite dimensional observability map on a finite interval $[0,t]$, $\mathcal{O} : H \to H([0,t], Y)$ is a bounded linear map defined as:

$$(\mathcal{O}z)(t) = CS(t)z \tag{9}$$

The observability gramian on the finite interval $[0,t]$ is given in operator form as $W_o(0,t) = \mathcal{O}^*\mathcal{O}$, where $\mathcal{O}^*$ is the adjoint operator of $\mathcal{O}$. The observability gramian on the time interval $[0,t]$, can be computed as an element of the linear space $\mathcal{L}(H,H)$ from the expression, see Curtain and Zwart (2020):

$$W_o(0,t)z = \int_0^t S^*(\tau)C^*CS(\tau)zd\tau. \tag{10}$$

From the derivations above follows, that the computation of infinite dimensional system gramians reduces to obtaining the solutions of the infinite dimensional abstract state and output equations.

## 3. GRAMIANS EVALUATION FOR THE VIBRATING STRING SYSTEM

The vibrating string system is a basic example for hyperbolic distributed parameter systems that is described by the wave partial differential equation. We undertake direct approach for deriving the solution for the wave PDE as shown in Farlow (1982), which is further used to compute the gramians for the vibrating string and beam systems.

The application of the law of Newton to a vibrating string of length $l$ leads to the following partial differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < l, \quad t \geq 0, \tag{11}$$

where $u(x,t)$ is the string deviation from the equilibrium position, $\alpha^2$ is a constant parameter and $f(x,t)$ is a continuous function representing the external force acting on the string. We assume that the string ends are fixed and the wave equation is with homogeneous boundary conditions: $u(0,t) = 0$ and $u(l,t) = 0$. Further assume continuous initial conditions: $u(x,0) = \varphi(x)$ and $\frac{\partial u}{\partial t}(x,0) = \psi(x)$. The vibrating string problem can be described in the framework of abstract differential equation (2), where matrix $A$ is in the form $A = \begin{pmatrix} 0 & 1 \\ -A_0 & 0 \end{pmatrix}$ where $A_0 h = -\frac{d^2h}{dx^2}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The domain of the operator $A_0$ is defined as, see Curtain and Zwart (2020):

$$D(A_0) = \{h \in L_2(0,l) : h, \frac{dh}{dx} - continuous, \\ \frac{d^2h}{dx^2} \in L_2(0,l); h(0) = 0 = h(l)\}$$

The domain of the operator $A$ is $D(A) = D(A_0^{\frac{1}{2}}) \times L_2(0,l)$, see Curtain and Zwart (2020). Applying the time space separation of variables method, see Li and Qi (2010), we look for the solution of equation (11) in the form:

$$u(x,t) = T(t)X(x) \tag{12}$$

Consider first the homogeneous equation when the external force acting on the string is zero, i.e. $f(x,t) = 0$. Then the differential equation takes the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < l, \quad t \geq 0, \tag{13}$$

with initial conditions $u(x,0) = \varphi(x)$ and $\frac{\partial u}{\partial t}(x,0) = \psi(x)$ and boundary conditions $u(0,t) = u(l,t) = 0$. Following the approach presented in Farlow (1982) and substituting (12) in (13), we obtain the following expression:

$$X(x)T^{''}(t) = \alpha^2 X^{''}(x)T(t) \tag{14}$$

which can be presented as:

$$\frac{1}{\alpha^2}\frac{T^{''}(t)}{T(t)} = \frac{X^{''}(x)}{X(x)} \tag{15}$$

Since both sides of (15) are independent from each other, they are equal to a constant $\lambda$. So, we obtain the equation:

$$X^{''}(x) - \lambda X(x) = 0 \tag{16}$$

with boundary conditions $X(0) = 0$ and $X(l) = 0$. For $\lambda \geq 0$, the solution of equation (16) reduces to the trivial solution $X(x) \equiv 0$, see Farlow (1982). The only nontrivial solution exists when $\lambda < 0$ and in this case, we obtain the solution in the form:

$$X(x) = C\sin\sqrt{-\lambda}x + D\cos\sqrt{-\lambda}x \tag{17}$$

Since $X(0) = 0$, From (17) is clear that $D = 0$. Since $X(l) = 0$, $C\sin\sqrt{-\lambda}l = 0$. The solution of this equation is $\sqrt{-\lambda} = \frac{n\pi}{l}$ or we obtain $\lambda_n = -(\frac{n\pi}{l})^2$, $n = 1,2,\ldots$. The values of $\lambda$ for which the equation (17)) has nontrivial solution are called eigenvalues of the wave equation (13) and the functions $\phi_n(x) = \sin\sqrt{-\lambda_n}x = \sin\frac{n\pi}{l}x$ are called eigenfunctions for the wave equation (13). Therefore, the wave equation has nontrivial solutions in terms of its eigenfunctions $X_n(x) = \phi_n(x) = \sin\frac{n\pi}{l}x$ for $n = 1,2,\ldots$.

For each eigenvalue $\lambda_n = -(\frac{n\pi}{l})^2$, there exists a function $T_n(t)$, which is a solution of the differential equation:

$$T_n^{''}(t) + \left(\frac{n\pi}{l}\right)^2 \alpha^2 T_n(t) = 0 \tag{18}$$

The solution of equation (18) can be determined as:

$$T_n(t) = A_n\cos\frac{\alpha n\pi}{l}t + B_n\sin\frac{\alpha n\pi}{l}t \tag{19}$$

The solution of (13) can be presented in the form $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, where:

$$u_n(x,t) = \left[A_n\cos\frac{\alpha n\pi}{l}t + B_n\sin\frac{\alpha n\pi}{l}t\right]\sin\frac{n\pi}{l}x, \tag{20}$$

Introducing the summation operator, we can write:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\frac{\alpha n\pi}{l}t + B_n\sin\frac{\alpha n\pi}{l}t\right]\sin\frac{n\pi}{l}x \tag{21}$$

The coefficients $A_n$, $B_n$, $n = 1,2,\ldots$ in (21) can be determined from the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{l}x = \varphi(x) \tag{22}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\frac{\alpha n\pi}{l}B_n\sin\frac{n\pi}{l}x = \psi(x) \tag{23}$$

for $0 < x < l$. The equations (22) and (23) can be considered as Fourier series presentations of the continuous functions $\varphi(x)$ and $\psi(x)$ with respect to the complete orthonormal system of functions $\phi_n(x) = \sin\frac{n\pi}{l}x$, $n = 1,2,\ldots$. Therefore, the coefficients $A_n$ and $\frac{\alpha n\pi}{l}B_n$ are the Fourier coefficients $\varphi_n$ and $\psi_n$ in the series expansion with respect to the system $\{\phi_n(x)\}$, $n = 1,2,\ldots$. These coefficients can be determined by the standard formulas:

$$A_n = \varphi_n = \langle\varphi,\phi_n\rangle = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}xdx \tag{24}$$

$$B_n = \frac{l\psi_n}{\alpha n\pi} = \frac{l\langle\psi,\phi_n\rangle}{\alpha n\pi} = \frac{2}{\alpha n\pi}\int_0^l \psi(x)\sin\frac{n\pi}{l}xdx \tag{25}$$

Therefore, the solution of the homogeneous wave equation can be presented in the form:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\langle\varphi,\phi_n\rangle\cos\frac{\alpha n\pi}{l}t + \langle\psi,\phi_n\rangle\frac{l}{\alpha n\pi}\sin\frac{\alpha n\pi}{l}t\right] \cdot \phi_n(x) \tag{26}$$

The second state variable in the abstract differential equation can be written as:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-\frac{\alpha n\pi}{l}\langle\varphi,\phi_n\rangle\sin\frac{\alpha n\pi}{l}t + \langle\psi,\phi_n\rangle\cos\frac{\alpha n\pi}{l}t\right] \cdot \phi_n(x) \tag{27}$$

We use the following notations:

$\omega_n = \frac{\alpha n\pi}{l}$, $z_n(t) = \begin{bmatrix} u_n(\cdot,t) \\ \dfrac{du_n(\cdot,t)}{dt} \end{bmatrix}$, $z_n(0) = \begin{bmatrix} \langle\varphi,\phi_n\rangle \\ \langle\psi,\phi_n\rangle \end{bmatrix}$.

Then,we can write the $n^{th}$ components of equations (26) and (27) as:

$$z_n(t) = \begin{bmatrix} \cos\omega_n t & \dfrac{1}{\omega_n}\sin\omega_n t \\ -\omega_n \sin\omega_n t & \cos\omega_n t \end{bmatrix} z_n(0)\cdot\phi_n \quad (28)$$

We consider the following matrix from equation (28):

$$\Lambda_n(t) = \begin{bmatrix} \cos\omega_n t & \dfrac{1}{\omega_n}\sin\omega_n t \\ -\omega_n \sin\omega_n t & \cos\omega_n t \end{bmatrix}, \quad (29)$$

It can be easily seen that the eigenvalues of $\Lambda_n(t)$ are $\cos(\omega_n t) \pm j\sin(\omega_n t)$ and therefore, after using the Euler's equality, we can claim that $\Lambda_n(t)$ is similarly equivalent to the matrix $diag\{e^{j\omega_n t}, e^{-j\omega_n t}\}$. It is obvious that matrix $\Lambda_n(t)$ satisfies the conditions for strongly continuous $C_0$ semigroup. Therefore, equation (28) can be considered as the $n^{th}$ component of equation (6), i.e. the Riesz-spectral operator representation of $z_n(t)$. Next we assume that the system output is the string deviation from the equilibrium, i.e. $y(t) = u(\cdot, t)$. We can write:

$$y(t) = \sum_{n=1}^{\infty} C_n \Lambda_n(t) z_n(0)\cdot\phi_n, \quad (30)$$

where $C_n = [\,1\ 0\,]$. Expression (30) can be written in matrix form as:

$$y(t) = \sum_{n=1}^{\infty} C_n \Lambda_n(t)\Phi_n z_n(0), \quad (31)$$

where $\Phi_n = \Phi_n(x)$ is a $[\,2\times 2\,]$ diagonal matrix $\Phi_n = \begin{bmatrix} \phi_n & 0 \\ 0 & \phi_n \end{bmatrix}$. The observability gramian is computed on the interval $[0, t]$ by implementing the equation (10) in the form:

$$W_o[(0,t), x] = \sum_{n=1}^{\infty}\left[\int_0^t \Lambda_n^T(\tau) C_n^T C_n \Lambda_n(\tau) d\tau\right]\Phi_n(x) \quad (32)$$

where $\Lambda_n(t)$ is defined as in (29) with $\omega_n = \frac{\alpha n\pi}{l}$, $B_n$ is a $[\,2\times 1\,]$ vector column $B_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\Phi_n(x) = \Phi_n$ is a $[\,2\times 2\,]$ diagonal matrix defined as $\Phi_n(x) = \begin{bmatrix} \phi_n(x) & 0 \\ 0 & \phi_n(x) \end{bmatrix}$.

For evaluating the controllability gramian for the vibrating string distributed parameter system, we consider the nonhomogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\ 0 < x < l,\ t \geq 0, \quad (33)$$

with zero boundary conditions $u(0,t) = u(l,t) = 0$ and zero initial conditions $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$. We use the time-space separation method to find the solution of (33) with homogeneous initial and boundary conditions, see Farlow (1982). We assume that the Fourier series is convergent on the finite interval $(0, l)$ and that the continuous function $f(x,t)$ can also be presented in terms of the complete orthonormal set of eigenfunctions $\phi_n(x) = \sin\frac{n\pi}{l}x$ as follows:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi}{l}x \quad (34)$$

where $f_n(t)$ are the Fourier series coefficients in the series expansion and can be computed from the expression:

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi}{l}x dx \quad (35)$$

Then, the equation (33) can be written in the form:

$$\sum_{n=1}^{\infty}\left[T_n^{''}(t) + \left(\frac{\alpha n\pi}{l}\right)^2 T_n(t) - f_n(t)\right]\sin\frac{n\pi}{l}x = 0 \quad (36)$$

Therefore, the coefficients of the Fourier series expansion (36) have to be all zero and the time-domain function function $T_n(t)$ satisfies the following equation:

$$T_n^{''}(t) + \left(\frac{\alpha n\pi}{l}\right)^2 T_n(t) = f_n(t),\ \ T_n(0) = T_n^{'}(0) = 0 \quad (37)$$

Equation (37) is an ordinary differential equation and its solution is given in the form:

$$T_n(t) = \frac{l}{\alpha n\pi}\int_0^t \sin\left[\frac{\alpha n\pi}{l}(t-\tau)\right] f_n(\tau) d\tau, \quad (38)$$

where expression (38) is the well known Duhamel integral for the convolution operator of linear systems. After substitution of (38) in the expression for $u(x,t)$, we find the solution of the nonhomogeneous wave equation under zero initial and boundary conditions as follows:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{l}{\alpha n\pi}\int_0^t \sin\left[\frac{\alpha n\pi}{l}(t-\tau)\right] f_n(\tau)d\tau\right] \cdot \sin\frac{n\pi}{l}x \quad (39)$$

The time derivative of $u(x,t)$ satisfies the following equation:

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left[\int_0^t \cos\frac{\alpha n\pi}{l}(t-\tau) f_n(\tau) d\tau\right]\cdot\sin\frac{n\pi}{l}x \quad (40)$$

Making the same notations as before, we obtain for the $n^{th}$ state vector component of the abstract differential equation the following expression:

$$z_n(t) = \int_0^t \begin{bmatrix} \cos\omega_n(t-\tau) & \dfrac{1}{\omega_n}\sin\omega_n(t-\tau) \\ -\omega_n\sin\omega_n(t-\tau) & \cos\omega_n(t-\tau) \end{bmatrix}\cdot B_n \upsilon_n(\tau) d\tau \cdot \sin\frac{n\pi}{l}x, \quad (41)$$

where $B_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\upsilon_n(t) = f_n(t)$. The state $z(t)$ containing $z_n(t)$ from (41) can be written in matrix form as:

$$z(t) = \sum_{n=1}^{\infty}\left[\int_0^t \Lambda_n(t-\tau)\Phi_n \Upsilon_n(\tau) d\tau\right], \quad (42)$$

where $\Upsilon_n(t)$ is a $[\,2\times 1\,]$ vector column $\Upsilon_n(t) = B_n\cdot\upsilon_n(t)$ and $\Phi_n = \Phi_n(x)$ is a $[\,2\times 2\,]$ diagonal matrix $\Phi_n = \begin{bmatrix} \phi_n & 0 \\ 0 & \phi_n \end{bmatrix}$.

Following the definition for the controllability gramian as in (8), we derive the following expression for the gramian:

$$W_c[(0,t),x] = \sum_{n=1}^{\infty}\left[\int_0^t \Lambda_n(\tau)B_nB_n^T\Lambda_n^T(\tau)d\tau\right]\Phi_n(x) \quad (43)$$

where $\Lambda_n(t)$ is defined as in (29) with $\omega_n = \frac{\alpha n\pi}{l}$, $B_n$ is a $[\,2\times 1\,]$ vector column $B_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\Phi_n(x) = \Phi_n$.

## 4. GRAMIANS EVALUATION FOR THE VIBRATING BEAM SYSTEM

We consider the partial differential equation describing the vibrating beam system:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -\alpha^2\frac{\partial^4 u(x,t)}{\partial x^4}, \quad 0 < x < l, \quad t \geq 0, \quad (44)$$

where $\alpha^2 = \frac{\kappa}{\rho}$ is the parameter with $\kappa$ - modulus of deflection, $\rho$ - material linear density and $f(x,t)$ is the external force acting on the beam. We assume zero boundary conditions: $u(0,t) = u(l,t) = 0$ and $u_{xx}(0,t) = u_{xx}(l,t) = 0$. The initial conditions for the problem are $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$, where $\varphi(x)$ and $\psi(x)$ are continuous functions. We apply the method of time-space variables separation and present the solution in the form $u(x,t) = T(t)X(x) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$.

Following the approach presented in Farlow (1982), we consider the homogeneous equation:

$$X(x)T^{''}(t) = -\alpha^2X^{(IV)}(x)T(t) \quad (45)$$

From (45) we obtain the following expression:

$$\frac{1}{\alpha^2}\frac{T^{''}(t)}{T(t)} = -\frac{X^{(IV)}(x)}{X(x)} = \lambda, \quad (46)$$

where $\lambda = -\omega^2$ is a negative number. From (46) we obtain the fourth order differential equation:

$$X^{(IV)} - \omega^2X(x) = 0 \quad (47)$$

Equation (47) has the following general solution, see Farlow (1982):

$$\begin{aligned} X(x) &= A\cos\sqrt{\omega}x + B\sin\sqrt{\omega}x \\ &\quad + C\sinh\sqrt{\omega}x + D\cosh\sqrt{\omega}x \end{aligned} \quad (48)$$

From the boundary conditions at $x = 0$, we have $X(0) = 0$ and $X^{''}(0) = 0$, therefore $A + D = 0$ and $-A + D = 0$. Thus, we obtain $A = D = 0$. From the boundary conditions at $x = l$, we have $X(l) = 0$ and $X^{''}(l) = 0$, therefore $B\sin\sqrt{\omega}l + C\sinh\sqrt{\omega}l = 0$ and $-B\sin\sqrt{\omega}l + C\sinh\sqrt{\omega}l = 0$. From the last two equations we obtain $Csinh\sqrt{\omega}l = 0$ and $B\sin\sqrt{\omega}l = 0$ and therefore, $C = 0$ and $\sqrt{\omega}l = n\pi$. The natural frequencies of the vibrating beam system are $\omega_n = \left(\frac{n\pi}{l}\right)^2$ and the eigenvalues are obtained as $\lambda_n = -\left(\frac{n\pi}{l}\right)^4$. The eigenfunctions of the problem are $X_n(x) = \phi_n(x) = \sin\frac{n\pi}{l}x$, $n = 1,2,\ldots$. Next, we consider the function $T_n(t)$, from the differential equation (46):

$$T_n^{''}(t) + \left(\frac{n\pi}{l}\right)^4\alpha^2T_n(t) = 0 \quad (49)$$

The solution of equation (49) takes the form:

$$T_n(t) = a_n\cos\left[\left(\frac{n\pi}{l}\right)^2\alpha t\right] + b_n\sin\left[\left(\frac{n\pi}{l}\right)^2\alpha t\right] \quad (50)$$

Finally, the solution of the homogeneous vibrating beam equation is obtained as follows:

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}\left[a_n\cos\left[\left(\frac{n\pi}{l}\right)^2\alpha t\right] + b_n\sin\left[\left(\frac{n\pi}{l}\right)^2\alpha t\right]\right] \\ &\cdot \sin\frac{n\pi}{l}x \end{aligned} \quad (51)$$

The coefficients $a_n$ and $b_n$, $n = 1,2,\ldots$ are obtained from the initial conditions: $u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi}{l}x$ and $u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty}\alpha\left(\frac{n\pi}{l}\right)^2 b_n\sin\frac{n\pi}{l}x$. Since the eigenfunctions $\phi_n(x) = \sin\frac{n\pi}{l}x$, $n = 1,2,\ldots$ form a complete orthonormal set in the Hilbert space $\mathcal{H}$, we can compute the coefficients $a_n$ and $b_n$, $n = 1,2,\ldots$ as coefficients in Fourier series expansions:

$$a_n = \varphi_n = \langle\varphi,\phi_n\rangle = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}xdx \quad (52)$$

$$\begin{aligned} b_n &= \frac{1}{\alpha}\left(\frac{l}{n\pi}\right)^2\psi_n = \frac{1}{\alpha}\left(\frac{l}{n\pi}\right)^2\langle\psi,\phi_n\rangle \\ &= \frac{2l}{\alpha(n\pi)^2}\int_0^l \psi(x)\sin\frac{n\pi}{l}xdx \end{aligned} \quad (53)$$

Therefore, the solution of the homogeneous vibrating beam equation is:

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}\langle\varphi,\phi_n\rangle\cos\left[\left(\frac{n\pi}{l}\right)^2\alpha t\right]\sin\frac{n\pi}{l}x \\ &+ \frac{1}{\alpha}\left(\frac{l}{n\pi}\right)^2\langle\psi,\phi_n\rangle\sin\left[\left(\frac{n\pi}{l}\right)^2\alpha t\right]\sin\frac{n\pi}{l}x \end{aligned} \quad (54)$$

The second state variable in the abstract diffrenetial equation can be written as:

$$\begin{aligned} \frac{\partial u}{\partial t} = &\sum_{n=1}^{\infty} -\alpha\left(\frac{n\pi}{l}\right)^2\langle\varphi,\phi_n\rangle\sin\left[\alpha\left(\frac{n\pi}{l}\right)^2 t\right]\sin\frac{n\pi}{l}x \\ &+ \langle\psi,\phi_n\rangle\cos\left[\alpha\left(\frac{n\pi}{l}\right)^2 t\right]\sin\frac{n\pi}{l}x \end{aligned} \quad (55)$$

We use the following notations:

$$\omega_n = \alpha\left(\tfrac{n\pi}{l}\right)^2,\ z_n(t) = \begin{bmatrix} u_n(\cdot,t) \\ \dfrac{du_n(\cdot,t)}{dt} \end{bmatrix},\ z_n(0) = \begin{bmatrix} \langle\varphi,\phi_n\rangle \\ \langle\psi,\phi_n\rangle \end{bmatrix}.$$

We build the matrix $\Lambda_n(t)$ as in (29) with input argument $\omega_n = \alpha\left(\frac{n\pi}{l}\right)^2$ and present the relation between $z_n(t)$ and $z_n(0)$ as in (28). Forming $C_n$ and $\Phi_n(x)$ as in the vibrating string system we obtain the relation (31) for the vibrating beam system. Finally, we obtain the observability gramian of the vibrating beam as in (32).

For computing the controllability gramian of the vibrating beam distributed parameter system, we consider the nonhomogenous equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -\alpha^2 \frac{\partial^4 u(x,t)}{\partial x^4} + f(x,t), \quad 0 < x < l, \quad t \geq 0, \tag{56}$$

with zero boundary conditions $u(0,t) = u(l,t) = 0$ and $u_{xx}(0,t) = u_{xx}(l,t) = 0$, and zero initial conditions $u(x,0) = u_t(x,0) = 0$. We look for the solutions of the nonhomogeneous wave equations by using the Fourier series method in the form $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x$.

We assume that the Fourier series is convergent and that the function $f(x,t)$ can also be presented in terms of Fourier series as $f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x$, where $f_n(t)$ are the Fourier series coefficients in the series expansion and can be computed from the expression $f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi}{l} x dx$. Then, the equation (56) can be written in the form:

$$\sum_{n=1}^{\infty} \left[ T_n^{''}(t) + \alpha^2 \left( \frac{n\pi}{l} \right)^4 T_n(t) - f_n(t) \right] \sin \frac{n\pi}{l} x = 0 \tag{57}$$

The solution for $T_n(t)$ from (57) takes the form:

$$T_n(t) = \frac{l^2}{\alpha (n\pi)^2} \int_0^t \sin \left[ \alpha \left( \frac{n\pi}{l} \right)^2 (t - \tau) \right] f_n(\tau) d\tau \tag{58}$$

Using the expressions for $\Lambda_n(t)$ in (29) with $\omega_n = \alpha \left( \frac{n\pi}{l} \right)^2$, we can write the solution to the vibrating beam problem into a matrix form as:

$$z(t) = \sum_{n=1}^{\infty} \left[ \int_0^t \Lambda_n(t-\tau) \Phi_n \Upsilon_n(\tau) d\tau \right] \tag{59}$$

where $\Phi_n = \Phi_n(x)$, $\Lambda_n(t)$ is defined as in (29) for the vibrating string case with $\omega_n = \alpha \left( \frac{n\pi}{l} \right)^2$, $\Upsilon_n(t) = \begin{bmatrix} 0 \\ \upsilon_n(t) \end{bmatrix}$ with $\upsilon_n(t) = f_n(t)$. Then, the controllability gramian takes the form as in (43) with $B_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

## 5. CONCLUSION

This paper considers the problem of controllability and observability gramians computation for certain linear, time-invariant hyperbolic distributed parameter systems. A direct approach for solving the partial differential equation is considered, and the solution is obtained by using time space variables separation and the Fourier series method. Two different distributed parameter systems are explored: the vibrating string and beam systems. The same approach for deriving the solutions in both cases is applied, which is based on obtaining the eigenvalues and eigenfunctions for the stated problems. The obtained results comply with the general theory for infinite dimensional systems, based on the Riesz-spectral operator interpretation of the state space description and deriving of $C_0$ strongly continuous semigroups in Hilbert spaces. The obtained gramians are easy to compute and require elementary operations to derive.


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