COSINE AND COTANGENT THEOREMS FOR A QUADRILATERAL, TWO NEW FORMULAS FOR ITS AREA AND THEIR APPLICATIONS

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Abstract. Here we show new relationships between elements of a convex quadrilateral. They generalize the cosine and the so-called cotangent theorem for a triangle. We named the new ones cosine and cotangent theorems for a quadrilateral. We derive by them new formulae for the area of any quadrilateral, which help to find various relationships in a triangle and a quadrilateral (a Carnot theorem for a triangle and the Brahmagupta's theorem for the area of an inscribed quadrilateral are generalized, as examples).

1. INTRODUCTION

In the last time there were discovered many noticeable points in an arbitrary convex quadrilateral (see the reference list at the end of this work). Part of them were defined analogously to some noticeable points of a triangle. It became clear, as we will see, that besides the properties of remarkable points, some popular theorems for a triangle can be transferred to a quadrilateral (as the so-called cosine and cotangent theorems). Via the obtained cosine and cotangent theorems for a quadrilateral we proved unknown till now formulas for it's area. As two applications of derived here formulas and dependencies, we generalize Carnot theorem for a triangle and Brahmagupta's theorem for calculating the area of an inscribed quadrilateral.

2. COSINE AND COTANGENT THEOREMS FOR A QUADRILATERAL.

Before we formulate and prove the cosine and the cotangent theorems for a quadrilateral, let us remind and prove the cotangent theorem for triangle, as it is less popular, and then use it.

Theorem 1 (Cotangent theorem for a triangle). The side lengths of $\triangle ABC$ are $AB = a$, $BC = b$ and $CA = c$, $\sphericalangle ACB = \gamma$ and S is the triangle's area. This relationship is valid:

$$c^2 = a^2 + b^2 - 4S.\cot\gamma \tag{1}$$

Proof 1. According to the cosine theorem for $\triangle ABC$ we have (**Fig. 1**):

$$c^2 = a^2 + b^2 - 2ab.\cos\gamma .$$

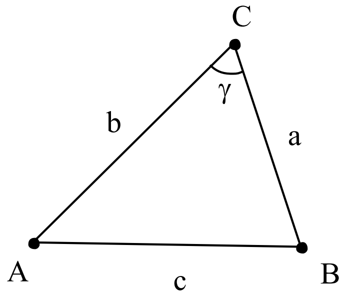

Figure 1. Shows objects from **Theorem 1**.

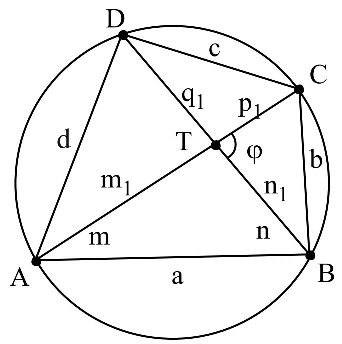

Figure 2. Shows objects from **Theorem 2, 3 and 4**.

From the other side, there holds the equation $S = \dfrac{1}{2}ab.\sin\gamma$. From here we derive:

$$c^2 = a^2 + b^2 - 4\frac{ab}{2}\cdot\sin\gamma.\cot\gamma = a^2 + b^2 - 4S.\cot\gamma .$$

Thus the equation (1) is proved.

Now we are able formulate and prove the cosine and cotangent theorems for a quadrilateral:

Theorem 2. (Cosine theorem for a quadrilateral). Denote the side lengths AB, BC, CD and DA in a quadrilateral $ABCD$ with a, b, c and d, m and n – the lengths of the diagonals AC and BD, and φ – the measure of the angle between the diagonals, opposite to BC (**Fig. 2**). Then:

$$b^2 + d^2 = a^2 + c^2 - 2mn.\cos\varphi \tag{2}$$

Proof 2. Let the diagonals AC, BD intersect at point T and $AT = m_1$, $BT = n_1$, $CT = p_1$, $DT = q_1$. Applying the cosine theorem to $\triangle ABT$, $\triangle BCT$, $\triangle CDT$ and $\triangle DAT$, we get respectively:

$$\begin{cases} a^2 = m_1^2 + n_1^2 - 2m_1n_1.\cos\left(180^\circ - \varphi\right) \\ b^2 = n_1^2 + p_1^2 - 2n_1p_1.\cos\varphi \\ c^2 = p_1^2 + q_1^2 - 2p_1q_1.\cos\left(180^\circ - \varphi\right) \\ d^2 = q_1^2 + m_1^2 - 2q_1m_1.\cos\varphi \end{cases} \qquad (3)$$

We add the first with the third equations of (3), and the second with the forth, and get:

$$a^2 + c^2 = m_1^2 + n_1^2 + p_1^2 + q_1^2 + 2m_1n_1.\cos\varphi + 2p_1q_1.\cos\varphi$$

$$b^2 + d^2 = n_1^2 + p_1^2 + q_1^2 + m_1^2 - 2n_1p_1.\cos\varphi - 2q_1m_1.\cos\varphi$$

From the last two ones there follows the equation:

$$a^2 + c^2 - 2m_1n_1.\cos\varphi - 2p_1q_1.\cos\varphi = b^2 + d^2 + 2n_1p_1.\cos\varphi + 2q_1m_1.\cos\varphi,$$

which can be transformed this way:

$$b^2 + d^2 = a^2 + c^2 - 2\left(m_1n_1 + n_1p_1 + p_1q_1 + q_1m_1\right).\cos\varphi.$$

As $m_1n_1 + n_1p_1 + p_1q_1 + q_1m_1 = \left(m_1 + p_1\right)\left(n_1 + q_1\right) = mn$, it leads to (2), which we wanted to prove.

Note 1: It's easy to guess, that in the boundary case, when the quadrilateral *ABCD* distorts in a $\triangle ABC$, i.e. when $D \to A$, then $d = 0$, $c = m$, $\varphi = \sphericalangle CAB$, $n = a$ and (2) gives then the relationship $b^2 = a^2 + m^2 - 2am.\cos\sphericalangle CAB$, which is the cosine theorem for $\triangle ABC$. This fact legitimates the usage of the term "cosine theorem" for this dependency.

The cotangent theorem for a quadrilateral is derived by the cosine theorem for it in the same way, as in the triangle.

Theorem 3 (Cotangent theorem for a quadrilateral). Let *ABCD* be a quadrilateral of side lengths *AB* = *a*, *BC* = *b*, *CD* = *c* and *DA* = *d*, and area *S*. If the angle between the diagonals, which is opposite to the side *BC*, is φ, then:

$$b^2 + d^2 = a^2 + c^2 - 4S.\cot\varphi \qquad (4)$$

Proof 3. Let the lengths of the diagonals *AC* and *BD*, be *m* and *n* resp. (**Fig. 2**). According to the proved cosine theorem for a quadrilateral, we have:

$$b^2 + d^2 = a^2 + c^2 - 2mn.\cos\varphi.$$

Therefore, having in mind the formula $S = \frac{1}{2}mn.\sin\varphi$ for a quadrilateral's area, we get:

$$b^2 + d^2 = a^2 + c^2 - 4\frac{mn}{2}\cdot\sin\varphi.\cot\varphi = a^2 + c^2 - 4S.\cot\varphi.$$

Thus the equation (4) is proved.

Note 2. It's easy to guess, that in the boundary case, when the quadrilateral *ABCD* becomes $\triangle ABC$, i.e. if $D \to A$, then $d = 0$, $c = m$, $\varphi = \sphericalangle CAB$ and the dependency (4) transforms to the dependency $b^2 = a^2 + m^2 - 4S.\cot\sphericalangle CAB$, i.e. in the cotangent theorem for the $\triangle ABC$. This legitimates the term "cotangent theorem", which we give to this relationship.

2. NEW FORMULAS FOR THE AREA OF AN ARBITRARY QUADRILATERAL.

From the proven relationship (4), which we've called cotangent theorem for a quadrilateral, in the case if $\varphi \neq 90^\circ$, the quadrilateral's area *S* can be expressed through the lengths of the sides and the tangent of the angle between the diagonals of the quadrilateral. We thus get the following unknown up to now formula for the area of an arbitrary quadrilateral:

$$S = \frac{1}{4}\left(a^2 + c^2 - b^2 - d^2\right).\tan\varphi, \quad \text{where } \varphi \neq 90^\circ \qquad (5)$$

Let us underline, that in this formula φ is those angle between the diagonals *AC* and *BD*, which lies opposite to the side of length *b*. With the help of the cosine theorem for a quadrilateral, a second new formula for its face is derived, by which it is expressed by the lengths of the sides and the diagonals of the quadrilateral.

Theorem 4. *ABCD* is a convex quadrilateral with side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$ and diagonals' lengths $AC = m$ and $BD = n$ (**Fig. 2**). The area S of the quadrilateral is expressed by these magnitudes through the formula:

$$S = \frac{1}{4}\sqrt{4m^2n^2 - \left(a^2 + c^2 - b^2 - d^2\right)^2} \qquad (6)$$

Proof 4. Via the cosine theorem for the quadrilateral *ABCD* we have:

$$b^2 + d^2 = a^2 + c^2 - 2mn.\cos\varphi .$$

$\Rightarrow \cos\varphi = \dfrac{a^2+c^2-b^2-d^2}{2mn}$, $\sin^2\varphi = \dfrac{4m^2n^2-\left(a^2+c^2-b^2-d^2\right)^2}{4m^2n^2}$ and as $S = \dfrac{1}{2}mn.\sin\varphi$, from the last equation we get (6), which we had to prove.

The just obtained formulas for area of a convex quadrilateral, and the cosine and the cotangent theorems for it, have important applications. With their help, for example, a series of inequalities connecting the lengths of the sides and the diagonals of any convex quadrilateral, as well as other important relationships between the lengths of the sides and the diagonals of the quadrilateral, are derived (see [1], [2] for more). Here we will apply the derived cotangent theorem for the quadrilateral and the second derived formula for its area to generalize two classical theorems of geometry.

3. A GENERALIZATION OF A CARNOT THEOREM.

The French engineer Lasar Carnot (1753 – 1823) has proved the following:

Theorem 5 (of Carnot). $\triangle ABC$ is an arbitrary one and l_1, l_2 and l_3 are the perpendiculars from arbitrary points A_1, B_1 and C_1 on *BC*, *CA* and *AB*, to the same sides (**Fig. 3**). The lines l_1, l_2 and l_3 meet at a single point if and only if there holds the equation:

$$CA_1^2 - BA_1^2 + AB_1^2 - CB_1^2 + BC_1^2 - AC_1^2 = 0 \qquad (7)$$

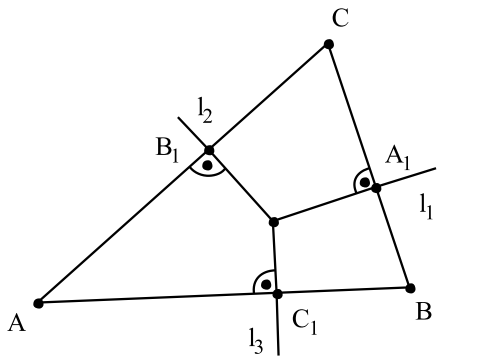

Figure 3. Shows objects in **Theorem 5.**

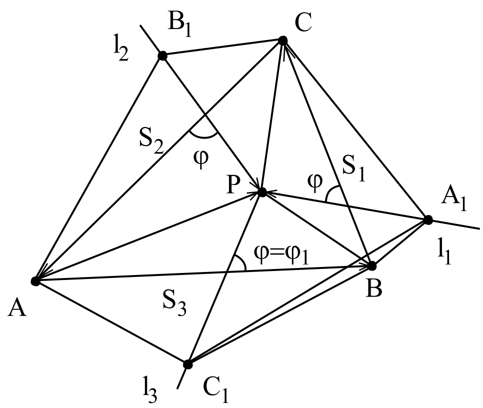

Figure 4. Shows objects in **Theorem 6.**

We will generalize the Theorem 5 by cancelling the condition the points A_1, B_1 and C_1 to lie either on the sides *BC*, *CA* and *AB* of $\triangle ABC$, or on the lines *BC*, *CA* and *AB*, and by replacing the perpendiculars l_1, l_2 and l_3 to these sides with lines, sloped to *BC*, *CA* and *AB* at the same angle φ:

Theorem 6 (Generalization of a Carnot theorem). $\triangle ABC$ is a positively oriented one (**Fig. 4**). A_1 is an arbitrary point either on the semi plane along the line *BC*, which do not include the triangle, or on the line *BC*. The points B_1 and C_1 satisfy the same conditions with respect to the lines *CA* and *AB*. The sloped lines $l_1^{\rightarrow}$, $l_2^{\rightarrow}$ and $l_3^{\rightarrow}$, respective to the sides *BC*, *CA* and *AB* of the $\triangle ABC$, pass through the points A_1, B_1 and C_1, and form angles of equal measure φ with the positive directions. If S is the area of the hexagon $AC_1BA_1CB_1$, which is not necessarily convex, then the lines l_1, l_2 and l_3 meet at a single point if and only if:

$$CA_1^2 - BA_1^2 + AB_1^2 - CB_1^2 + BC_1^2 - AC_1^2 = 4S.\cot\varphi \qquad (8)$$

Proof 6. 1) Let firs assume that the $l_1^{\to}$, $l_2^{\to}$ и $l_3^{\to}$ meet at a single point P (**Fig. 4**).
Denote S_1, S_2, S_3 the areas of the covering quadrilaterals BA_1CP, CB_1AP, AC_1BP of the hexagon $BA_1CB_1AC_1$. Via the cotangent theorem we get from these quadrilaterals resp.:

$$BA_1^2+CP^2=CA_1^2+BP^2-4S_1.\cot\varphi,$$
$$CB_1^2+AP^2=AB_1^2+CP^2-4S_2.\cot\varphi,$$
$$AC_1^2+BP^2=BC_1^2+AP^2-4S_3.\cot\varphi.$$

We add the last equalities term-by-term; as $S_1+S_2+S_3=S$, therefore:

$$BA_1^2+CP^2+CB_1^2+AP^2+AC_1^2+BP^2=CA_1^2+BP^2+AB_1^2+CP^2+BC_1^2+AP^2-4S.\cot\varphi$$

The last equation is easily simplified to (8). Thus we proved, that *if* the lines $l_1^{\to}$, $l_2^{\to}$ and $l_3^{\to}$, sloped at angle φ, meet at one point, *then* (8) holds.
2) Now we'll prove the inverse implication, i.e. that if (8) is true, then the lines $l_1^{\to}$, $l_2^{\to}$ and $l_3^{\to}$, sloped at angle φ, meet at one point. Denote P the common point of $l_1^{\to}$ and $l_2^{\to}$. It's sufficient to prove that the ray $C_1P^{\to}$ coincides with $l_3^{\to}$, which passes through the point C_1 under angle φ, i.e. that it forms angle φ with the positive direction of the side AB. Otherwise, we have to prove, that the angle in the quadrilateral AC_1BP, which form the diagonals C_1P and AB, which lies opposing the side BP, equals φ. Denote this angle φ_1. According to the cotangent theorem, from the quadrilaterals BA_1CP, CB_1AP and AC_1BP, we get respectively:

$$BA_1^2+CP^2=CA_1^2+BP^2-4S_1.\cot\varphi,$$
$$CB_1^2+AP^2=AB_1^2+CP^2-4S_2.\cot\varphi,$$
$$AC_1^2+BP^2=BC_1^2+AP^2-4S_3.\cot\varphi_1.$$

Adding these equations term by term, we get:

$$\left(BA_1^2+CP^2\right)+\left(CB_1^2+AP^2\right)+\left(AC_1^2+BP^2\right)=$$
$$=\left(CA_1^2+BP^2\right)+\left(AB_1^2+CP^2\right)+\left(BC_1^2+AP^2\right)-4\left(S_1+S_2\right).\cot\varphi-4S_3.\cot\varphi_1$$

and after simplification:

$$CA_1^2-BA_1^2+AB_1^2-CB_1^2+BC_1^2-AC_1^2=4\left(S_1+S_2\right).\cot\varphi+4S_3.\cot\varphi_1.$$

From the other side, we assume that (8) holds, which can be represented thus:

$$CA_1^2-BA_1^2+AB_1^2-CB_1^2+BC_1^2-AC_1^2=4\left(S_1+S_2\right).\cot\varphi+4S_3.\cot\varphi.$$

From the last two equations $4S_3.\cot\varphi_1=4S_3.\cot\varphi$, i.e. $\varphi=\varphi_1$, and the theorem is proved.

4. A GENERALIZATION OF BRAHMAGUPTA'S THEOREM.

From the new formula for area of a convex quadrilateral (formula (6)) we get in particular the famous Brahmagupta's formula (7 century AD) for area of an inscribed quadrilateral. As for such quadrilateral we have $mn=ac+bd$ (according to the Ptolemy theorem), after replacing in (6) we get (fig. 2):

$$S=\frac{1}{4}\sqrt{4\left(ac+bd\right)^2-\left(a^2+c^2-b^2-d^2\right)^2}=$$
$$=\frac{1}{4}\sqrt{\left(2ac+2bd+a^2+c^2-b^2-d^2\right)\left(2ac+2bd+b^2+d^2-a^2-c^2\right)}=$$
$$=\frac{1}{4}\sqrt{\left[\left(a+c\right)^2-\left(b-d\right)^2\right]\left[\left(b+d\right)^2-\left(a-c\right)^2\right]}=$$
$$=\frac{1}{4}\sqrt{\left(a+c+b-d\right)\left(a+c-b+d\right)\left(b+d+a-c\right)\left(b+d-a+c\right)}$$

By setting $p=\dfrac{a+b+c+d}{2}$, we get $S=\sqrt{(p-a)(p-b)(p-c)(p-d)}$, which is the Brahmagupta's formula for the area of an inscribed quadrilateral.
We see, that formula (8) for the area of a convex quadrilateral generalizes the Brahmagupta's formula for the area of an inscribed quadrilateral.

5. CONCLUSIONS.

The above-proven dependencies in an arbitrary quadrilateral and the formulas for its area serve to derive various other relationships in it. We will consider them in further articles.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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