New Formulas for Distances Between New and Traditional Remarkable Points in a Quadrilateral

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Abstract. Here we consider new geometrical objects and their properties, obtained in our previous works and several theorems, which provide *new* formulas for distances between *new* and traditional remarkable points in a quadrilateral, and other *new* relationships, namely: 1) the distances from (the intersection points of the perpendicular bisectors of the opposite sides) to the (vertices and the point of Miquel of the quadrilateral); and 2) the relationships between the: (side lengths and the measures of the angles between adjacent and opposite quadrilateral's sides), (distances between the intersection point of the perpendicular bisectors of the diagonals and the quadrilateral's vertices *with* the side lengths and the measures of the angles between the two diagonals and between each diagonal with each of it's adjacent sides), (side lengths and the angles between the two diagonals and between each diagonal and each its adjacent side) and (distances from the diagonals' intersection point to the Brocard points *with* the side-lengths and the midline segments).

INTRODUCTION

One of the most easily defined remarkable points in the quadrilateral is the intersection point of the bisectors of each two opposite sides and the intersection of the bisectors of the diagonals. They are characterized by a number of interesting properties and are closely related to some of the other remarkable points in the quadrilateral. For brevity, we will call the intersection P_1 of the perpendicular bisectors of the sides AB and CD of a quadrilateral $ABCD$ (Fig. 1) a bysector point corresponding to AB and CD, and the intersection P_2 of the perpendicular bisectors of the sides AD and BC – a bysector point corresponding to AD and BC. The intersection point P_3 of the bisectors of the diagonals AC and BD we call a bysector point corresponding to the diagonals. We guarantee the existence and uniqueness of the all bysector points, assumming the quadrilateral $ABCD$ has no two parallel sides. As we will see, there are simple formulas for the distances from the bysector points P_1, P_2 and P_3 to the vertices of $ABCD$, to its Miquel and Brocard points. We will summarize the properties of these points, which we will use.

1. PROPERTIES OF THE BROCARD POINTS AND THE MIQUEL POINT

Let *ABCD* be a convex quadrilateral and *T* be the point of intersection of its diagonals (Fig. 2).

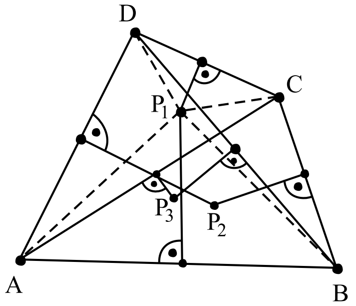
Definition 1. The second common point K_1 of the circumcircles of $\triangle ABT$ and $\triangle CDT$ is called a Brocard point of *ABCD*, corresponding to its sides *AB* and *CD*.

The Brocard point K_2, corresponding to *AD* and *BC*, is analogously defined. K_1 and K_2 have these properties:

Property 1. Form similar triangles with their resp. sides: $\triangle AK_1B \sim \triangle CK_1D$, $\triangle AK_2D \sim \triangle CK_2B$ (Fig. 2) ([1]).

Definition 2. The circle (*c*) through the midpoints *E* and *F* of the diagonals and their intersection point *T* is called a Brocard circle of the quadrilateral (Fig. 3).

Property 2. The Brocard points K_1 and K_2 lie on the Brocard circle of the quadrilateral ([2]).



Quadrilateral's: **FIGURE 1.** Bisector points.

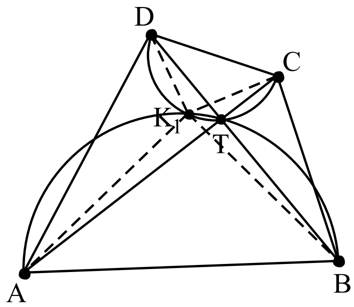


FIGURE 2. Brocardians.

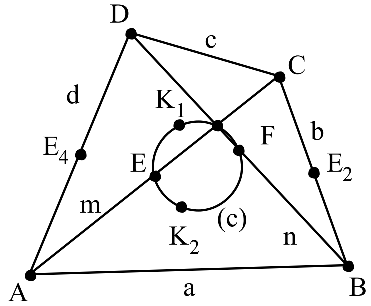


FIGURE 3. Brocard circle.

Property 3. Let *ABCD* be a convex quadrilateral and E_1, E_2, E_3, E_4 be the midpoints of its sides *AB*, *BC*, *CD*, *DA*, and *a*, *в*, *c*, *d* – their lengths. If *m*, *n* are the lengths of the diagonals *AC*, *BD*, and $E_2E_4 = l_1$ (Fig. 3), then:

$$[2]:\quad K_1A = \frac{ma}{2l_1},\quad K_1B = \frac{na}{2l_1},\quad K_1C = \frac{mc}{2l_1},\quad K_1D = \frac{nc}{2l_1}.$$

Let us recall the ***Definition of a Miquel point***: The extended sides *AD* and *BC* of a convex quadrilateral *ABCD* intersect at a point *U*, and the extended sides *AB* and *DC* – at a point *V*. The circumcircles of the $\triangle ABU$, $\triangle DCU$, $\triangle ADV$ and $\triangle BCV$ meet at a point *M*. It is called a Miquel point of the quadrilateral (Fig. 4).

The Miquel point of a quadrilateral is characterized by the following properties:

Property 4. *ABCD* is a convex quadrilateral, *E* and *F* – midpoints of it's diagonals *AC* and *BD*, $|EF| = l_3$ and the side's lengths *AB*, *BC*, *CD*, *DA* are *a*, *b*, *c*, *d*. The distances from the Miquel point *M* to its vertices are ([2]):

$$MA = \frac{ad}{2l_3},\quad MB = \frac{ab}{2l_3},\quad MC = \frac{bc}{2l_3},\quad MD = \frac{cd}{2l_3}.$$

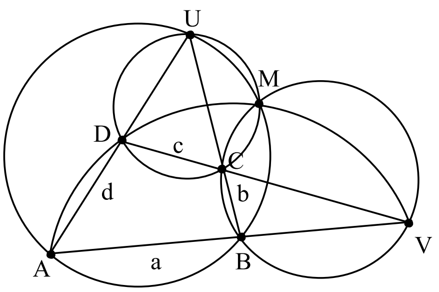


FIGURE 4. Miquel point of a quadrilateral.

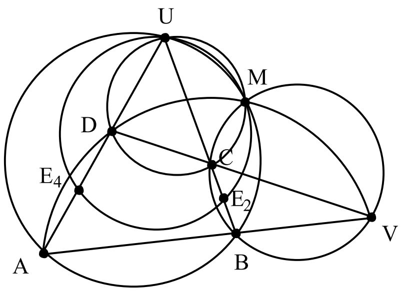


FIGURE 5. Miquel point *M* on the Brocard circles to its sides.

Definition 3. *ABCD* is a convex quadrilateral, the extensions of its sides *AD* and *BC* intersect at the point *U* (Fig. 5). If E_2 и E_4 are the midpoints of *AD* and *BC*, then the circle through E_2, E_4 and *U* we call a Brocard circle of *ABCD*, corresponding to the sides *AD* and *BC*. A Brocard circle, corresponding to *AB* and *CD*, is similarly defined.

Property 5. The Miquel point *M* of a quadrilateral lies on the Brocard circles corresp. to its sides (Fig. 5) ([3]).

2. FORMULAS FOR THE DISTANCES FROM THE QUADRILATERAL'S VERTICES TO THE INTERSECTIONS OF PERPENDICULAR BISECTORS' OPPOSITE SIDES

Theorem 1. *ABCD* is a convex quadrilateral, the extensions of its sides *AD* and *BC* intersect at a point *U*, $\sphericalangle AUB = \varphi$ (Fig. 6^a). The lengths of the sides *AB*, *BC*, *CD*, *DA* are *a*, *b*, *c*, *d*, the measures of the angles at *A*, *B*, *C*, *D* – *α*, *β*, *γ*, *δ*. If P_2 is the intersection of the perpendicular bisectors of *AD* and *BC*, then it follows that

$$P_2A = P_2D = \frac{1}{2\sin\varphi}\cdot\sqrt{a^2+c^2-2ac.\cos(\gamma-\beta)},\ P_2B = P_2C = \frac{1}{2\sin\varphi}\cdot\sqrt{a^2+c^2-2ac.\cos(\alpha-\delta)} \quad (1)$$

Proof: If *D* is between *A* and *U* (Fig. 6^a), $P_2A = P_2D = x$ and D_1 is such that $AD_1 \| BC$ and $\sphericalangle D_1CB = \sphericalangle ABC = \beta$, then it follows for the isosceles trapezoid *ABCD* that $CD_1 = AB = a$ and that P_2 lies on the perpendicular bisector of its base *BC*, which is a perpendicular bisector of AD_1 too. Hence $P_2D_1 = P_2A = x$. From $P_2D_1 = P_2D = P_2A = x$ follows that *A*, D_1, *D* lie on a circle (*c*) of center P_2 and radius *x*, i.e. that ΔD_1AD is inscribed in (*c*), and $\sphericalangle D_1AD = \sphericalangle AUB = \varphi$ (as $D_1A \| BC$). From ΔD_1AD, by the sine rule, $D_1D = 2x.\sin\varphi$.

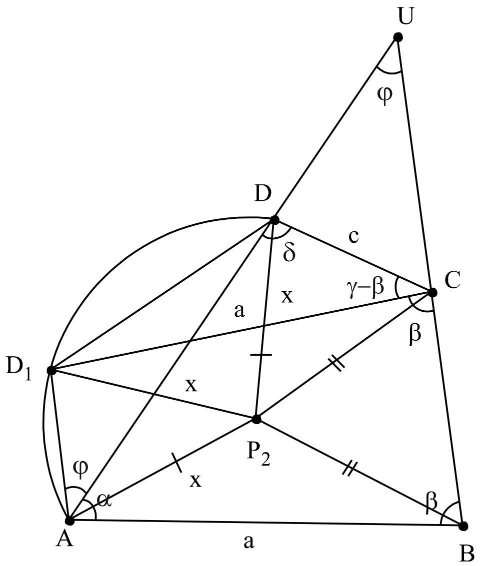


FIGURE 6^a.

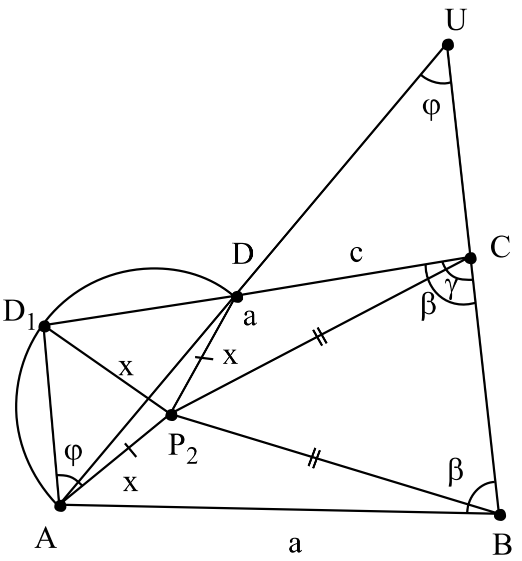


FIGURE 6^b.

There are three possible cases for the position of the point D_1 – it may lie on:

1) the half-plane with contour the line *CD* which contains the vertex *B* (fig. 6^a). In this case we have: $\sphericalangle D_1CD = \sphericalangle DCB - \sphericalangle D_1CB = \sphericalangle DCB - \sphericalangle ABC = \gamma - \beta$ and by the cosine rule from the $\Delta D_1DC \Rightarrow$
$\Rightarrow D_1D = \sqrt{D_1C^2 + DC^2 - 2D_1C.DC\cos\sphericalangle D_1CD} = \sqrt{a^2+c^2-2ac.\cos(\gamma-\beta)}$;

2) the half-plane with contour *CD*, which does not contain *B*. In this case the expression for DD_1 is the same;

3) *CD* (Fig. 6^b). Then $D_1D = |D_1C - DC| = |a-c|$, which $\Leftrightarrow D_1D = \sqrt{a^2+c^2-2ac\cdot\cos(\gamma-\beta)}$, because $\gamma-\beta = 0$ and $\cos(\gamma-\beta) = 1$. In all three cases $D_1D = \sqrt{a^2+c^2-2ac\cdot\cos(\gamma-\beta)}$. From this and from $D_1D = 2x.\sin\varphi \Rightarrow P_2A = P_2D = x = \frac{D_1D}{2\sin\varphi} = \frac{1}{2\sin\varphi}\cdot\sqrt{a^2+c^2-2ac\cdot\cos(\gamma-\beta)}$, which is the first of the equations (1) which we prove now. The second equation can be similarly proved.

Note. Analogously, the following formulas are obtained for the distances from the intersection P_1 of the perpendicular bisectors of the sides *AB* and *CD* to the vertices of the quadrilateral:

$$P_1A = P_1B = \frac{1}{2\sin\psi}\cdot\sqrt{b^2+d^2-2bd\cdot\cos(\delta-\gamma)},$$
$$P_1D = P_1C = \frac{1}{2\sin\psi}\sqrt{b^2+d^2-2bd\cdot\cos(\alpha-\beta)}$$

where $\psi = \sphericalangle\langle AB; DC\rangle$.

3. FORMULAS FOR THE DISTANCES FROM THE INTERSECTIONS OF THE PERPENDICULAR BISECTORS OF OPPOSITE SIDES TO THE MIQUEL POINT

Lemma. *ABCD* is a convex quadrilateral. The extensions of *AD* and *BC* meet at a point *U* and those of the sides *AB* and *DC* – at a point *V* (*C* lies between *D* and *V* and between *B* and *U*). Let the sides lengths *AB*, *BC*, *CD*, *DA* be *a*, *b*, *c* , *d*, and the measures of the angles at *A* and *D* – α and δ . If $\sphericalangle AUB = \varphi$, this equality holds (Fig. 7):

$$a \cdot \sin\alpha - c \cdot \sin\delta = b \cdot \sin\varphi. \tag{2}$$

Proof: Let *T* be the intersection of the diagonals *AC* and *BD*, and let $\sphericalangle AVD = \psi$ (Fig. 7). We have then:

$$a \cdot \sin\alpha - c \cdot \sin\delta = \frac{2}{d} \cdot \left(\frac{ad \cdot \sin\alpha}{2} - \frac{cd \cdot \sin\delta}{2} \right) = \frac{2}{d} \cdot \left(S_{ABD} - S_{ACD} \right) = \frac{2}{d} \cdot (S_{ABT} - S_{CDT}) =$$

$$= \frac{2}{d} \cdot (S_{ACV} - S_{BDV}) = \frac{2}{d} \cdot \left[\frac{CV \cdot (a + BV) \cdot \sin\psi}{2} - \frac{BV \cdot (c + CV) \cdot \sin\psi}{2} \right],$$

i.e.:

$$a \cdot \sin\alpha - c \cdot \sin\delta = \frac{\sin\psi}{d} \cdot (a \cdot CV - c \cdot BV). \tag{3}$$

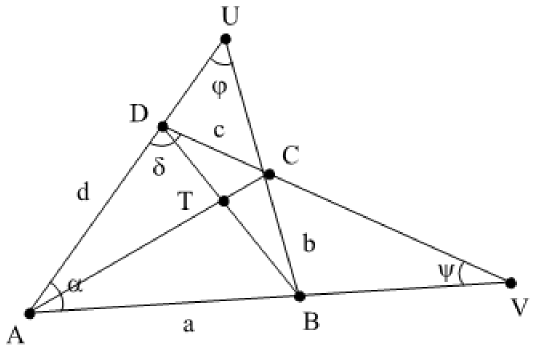


FIGURE 7. Lemma proof.

Further: from the $\triangle BCV$ and $\triangle DCU$, by the sine rule: $\frac{\sin\psi}{b} = \frac{\sin\sphericalangle BCV}{BV}$ and $\frac{\sin\varphi}{c} = \frac{\sin\sphericalangle UCD}{UD}$,

i.e.: $\sin\sphericalangle UCD = \frac{UD}{c} \cdot \sin\varphi$.

As $\sphericalangle BCV = \sphericalangle UCD$, we define: $\sin\psi = \frac{b.\sin\sphericalangle BCV}{BV} = \frac{b.\sin\sphericalangle UCD}{BV} = \frac{b}{BV} \cdot \frac{UD}{c} \cdot \sin\varphi$

Substituting in (3) we obtain:

$$a \cdot \sin\alpha - c \cdot \sin\delta = \frac{UD \cdot (a \cdot CV - c \cdot BV)}{BV \cdot dc} \cdot b \cdot \sin\varphi. \tag{4}$$

On the other hand, according to Menelaus' theorem for $\triangle AVD$ and the line *UCB* we have:

$$\frac{AB}{BV} \cdot \frac{CV}{DC} \cdot \frac{UD}{AU} = 1.$$

As *AB* = *a*, *DC* = *c*, *AU* = *d* + *UD*, hence we get: $a \cdot CV \cdot UD = BV \cdot c \cdot (d + UD)$,

i.e.:

$$UD \cdot (a \cdot CV - c \cdot BV) = BV \cdot dc.$$

From (4) and the last equality, it follows that $a \cdot \sin\alpha - c \cdot \sin\delta = \frac{UD \cdot (a \cdot CV - c \cdot BV)}{BV \cdot dc} \cdot b \cdot \sin\varphi = b \cdot \sin\varphi.$

Thus (2) is proved.

Theorem 2. *ABCD* is a convex quadrilateral, the extensions of *AD* and *BC* meet at the point *U*, $\sphericalangle AUB = \varphi$, the lengths of *AB* and *DC* are *a* and *c*, and the distance between the midpoints of the diagonals is l_3. The distance between the intersection P_2 of the perpendicular bisectors of *AD* and *BC*, and the Miquel point *M* (Fig. 8) is:

$$P_2M = \frac{\left| a^2 - c^2 \right|}{4l_3 \cdot \sin\varphi}. \tag{5}$$

Proof: Let the intersection of the extended sides AB and DC be V, $\sphericalangle AVD = \psi$ (Fig. 8). We'll consider only the case when C lies between D and V, and between U and B. If the midpoints of BC and AD are E_2 and E_4, then $\sphericalangle P_2E_4U = \sphericalangle P_2E_2U = 90°$. Therefore P_2, E_2, U and E_4 lie on a circle (k) with diameter P_2U. As E_2, E_4, U lie on (k), it is the Brocard circle, corresponding to AD and BC (by definition 3). Therefore, from property 5, $M \in (k)$, from where we get $\sphericalangle UMP_2 = 90°$. Miquel 's point M lies on ΔDCU's circumcircle (by definition). From the inscribed quadrilateral $DCMU$: $\sphericalangle UMC = \sphericalangle CDA = \delta$. Let $\overrightarrow{MP_2}$ lies between $\overrightarrow{MU}$ and $\overrightarrow{MC}$, from the last equations $\sphericalangle P_2MC = \sphericalangle UMC - \sphericalangle UMP_2 = \delta - 90°$. As M lies on the circumcircle of ΔBCV (by definition), and then $\sphericalangle BMC = \sphericalangle BVC = \psi$, therefore $\sphericalangle P_2MB = \sphericalangle P_2MC + \sphericalangle BMC = (\delta - 90°) + \psi$. The sum of the angles in the $\Delta AVD = 180° \Rightarrow (\delta - 90°) + \psi = 90 - \alpha \Rightarrow \sphericalangle P_2MB = 90° - \alpha$. By the cosine rule applied to the ΔP_2MC and ΔP_2MB, we obtain:

$$P_2C^2 = P_2M^2 + MC^2 - 2P_2M.MC.\cos\sphericalangle P_2MC$$
$$P_2B^2 = P_2M^2 + MB^2 - 2P_2M.MB.\cos\sphericalangle P_2MB$$

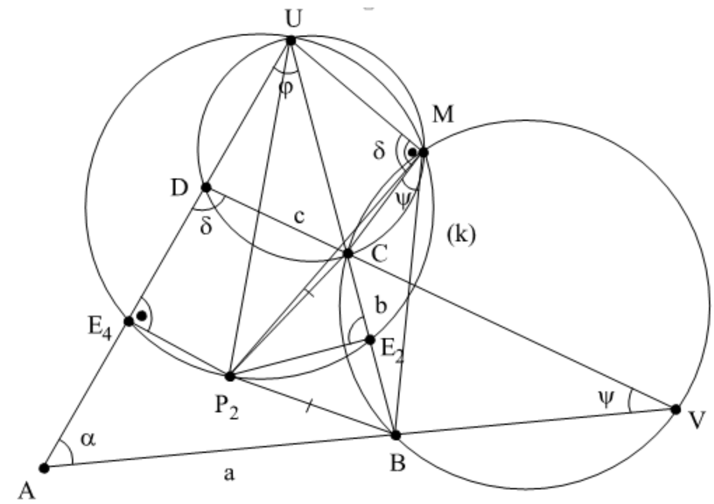


FIGURE 8. Theorem 2 Proof.

Since $P_2C = P_2B$, with subtraction of the last two equations, we find:

$$MB^2 - MC^2 = 2P_2M.MB.\cos\sphericalangle P_2MB - 2P_2M.MC.\cos\sphericalangle P_2MC \qquad (6)$$

But $MB = \dfrac{ab}{2l_3}$, $MC = \dfrac{cb}{2l_3}$ (from the property 4), $\sphericalangle P_2MB = 90° - \alpha$ and $\sphericalangle P_2MC = \delta - 90°$ (from above). We substitute these last four equalities in (6) and we get:

$$\frac{a^2b^2}{4l_3^{\,2}} - \frac{c^2b^2}{4l_3^{\,2}} = 2P_2M \cdot \frac{ab}{2l_3} \cdot \sin\alpha - 2P_2M \cdot \frac{cb}{2l_3} \cdot \sin\delta.$$

On the other hand $a \cdot \sin\alpha - c \cdot \sin\delta = b \cdot \sin\varphi$ (according to the lemma), and then from the last equation:

$$\frac{a^2b^2}{4l_3^{\,2}} - \frac{c^2b^2}{4l_3^{\,2}} = P_2M \cdot (a \cdot \sin\alpha - c \cdot \sin\delta) \cdot \frac{b}{l_3} = P_2M \cdot \frac{b^2 \sin\varphi}{l_3}.$$

From this immediately follows the equality (5) which we are proving now.

Note: The formula $P_1M = \dfrac{|b^2 - d^2|}{4l_3 \cdot \sin\psi}$ can be analogously proved (P_1 is the intersection of the perpendicular bisectors of the sides AB and CD).

4. FORMULAS FOR THE DISTANCES FROM THE POINT OF INTERSECTION P_3 OF THE PERPENDICULAR BISECTORS OF THE DIAGONALS TO THE VERTICES OF THE QUADRILATERAL AND TO ITS BROCARD POINTS K_1, K_2

Let's first derive formulas for the distances from the intersection point P_3 of the perpendicular bisectors of the diagonals to the vertices of the quadrilateral. Let's denote $\sphericalangle CAD = \alpha_1$, $\sphericalangle CAB = \alpha_2$, $\sphericalangle ABD = \beta_1$, $\sphericalangle DBC = \beta_2$, $\sphericalangle BDA = \delta_2$, $\sphericalangle ACB = \gamma_1$, $\sphericalangle ACD = \gamma_2$, $\sphericalangle BDC = \delta_1$ (Fig. 9).

Theorem 3. *ABCD* is a convex quadrilateral, *T* is the intersection point of its diagonals and $\sphericalangle ATB = \varphi_0$ (Fig. 10). The side lengths *AB*, *BC*, *CD*, *DA* are *a*, *b*, *c*, *d*. If P_3 is the intersection of the perpendicular bisectors of the diagonals *AC* and *BD*, then:

$$\begin{aligned} AP_3 = CP_3 &= \frac{1}{2\sin\varphi_0}\sqrt{a^2+c^2-2ac.\cos(\beta_1+\delta_1)} = \frac{1}{2\sin\varphi_0}\sqrt{b^2+d^2-2bd.\cos(\beta_2+\delta_2)} \\ BP_3 = DP_3 &= \frac{1}{2\sin\varphi_0}\sqrt{a^2+c^2-2ac.\cos(\alpha_2+\gamma_2)} = \frac{1}{2\sin\varphi_0}\sqrt{b^2+d^2-2bd.\cos(\alpha_1+\gamma_1)} \end{aligned} \qquad (7)$$

Proof: Let $AP_3 = CP_3 = x$ (Fig. 10) and C_1 be such a point, that $CC_1 \parallel BD$ and $\sphericalangle C_1BD = \sphericalangle CDB = \delta_1$. Then BC_1CD is an isosceles trapezoid, hence $BC_1 = CD = c$. The point P_3 is on the perpendicular bisector of the base *BD*, which is a perpendicular bisector of the base C_1C as well $\Rightarrow P_3C_1 = P_3C = x$. From $P_3A = P_3C = P_3C_1 = x \Rightarrow$ *A*, *C* and C_1 lie on the circle (*c*) with center P_3 and radius *x*, $\triangle ACC_1$ is inscribed in (c), so by the sine theorem $AC_1 = 2x.\sin\sphericalangle ACC_1$. But $\sphericalangle ACC_1 = \sphericalangle ATB = \varphi_0$ (as $BD \parallel CC_1$ by construction)

$$\Rightarrow AC_1 = 2x.\sin\varphi_0. \qquad (*)$$

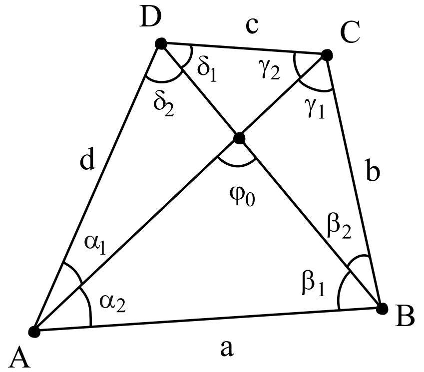


FIGURE 9. Notations for the next figure.

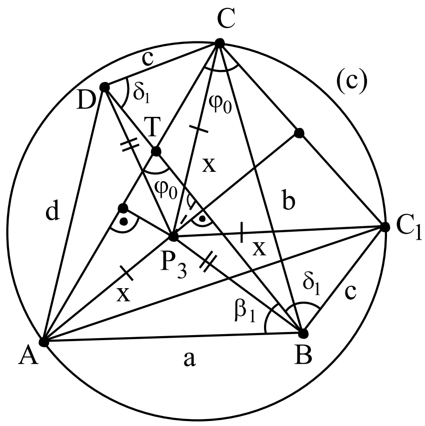


FIGURE 10. Theorem 3 Proof.

From the $\triangle ABC_1$, by the cosine theorem, we have:

$$AC_1 = \sqrt{AB^2 + C_1B^2 - 2|AB|.|C_1B|.\cos\sphericalangle ABC_1} \qquad (8)$$

Since $\sphericalangle ABC_1 = \sphericalangle ABD + \sphericalangle DBC_1 = \beta_1 + \sphericalangle BDC = \beta_1 + \delta_1$, $AB = a$ and $BC_1 = c$, from (8) we get $|AC_1| = \sqrt{a^2+c^2-2ac.\cos(\beta_1+\delta_1)}$. From the last and (*) we get the expression for AP_3 and CP_3 in (7): $|AP_3| = |CP_3| = x = \dfrac{|AC_1|}{2\sin\varphi_0} = \dfrac{\sqrt{a^2+c^2-2ac.\cos(\beta_1+\delta_1)}}{2\sin\varphi_0}$. Similarly, the second expression is obtained.

Then directly from the first of the proven equations (7) we derive:

Corollary: Every convex quadrilateral *ABCD* satisfies the identity:

$$a^2+c^2-2ac.\cos(\beta_1+\delta_1) = b^2+d^2-2bd.\cos(\beta_2+\delta_2)$$

We will now get formulas for the distances from the intersection P_3 of the perpendicular bisectors of the diagonals *AC* and *BD* of a convex quadrilateral to its Brocard points K_1, K_2.

Theorem 4. Let *ABCD* be a convex quadrilateral with intersection point *T* of the diagonals *AC* and *BD* and $\sphericalangle BTC = \varphi_0$ (Fig. 11). If the side lengths *AB*, *BC*, *CD*, *DA* are *a*, *b*, *c*, *d* and $E_1,\ E_2,\ E_3,\ E_4$ – the midpoints of *AB*, *BC*, *CD*, *DA*. If $E_2E_4 = l_1,\ E_1E_3 = l_2$, then the distances from P_3 to the Brocard points K_1 and K_2 are:

$$K_1P_3 = \frac{\left|a^2 - c^2\right|}{4l_1 \cdot \sin\varphi_0}, \quad K_2P_3 = \frac{\left|b^2 - d^2\right|}{4l_2 \cdot \sin\varphi_0}. \tag{9}$$

Proof: Let the midpoints of the diagonals *AC* and *BD* be *E* and *F*, the intersection of the extensions of the sides *AB* and *DC* be *V* and let $\sphericalangle AVD = \psi$ (Fig. 11). Since $\sphericalangle P_3ET = \sphericalangle P_3FT = 90^\circ$, the quadrilateral EP_3FT is inscribed in a circle (*c*) of diameter P_3T. I.e., P_3 lies on the circle through the points *E*, *F* and *T*, i.e. on the Brocard circle of *ABCD* (by definition 2). From property 2 $\Rightarrow K_1 \in (c) \Rightarrow \sphericalangle P_3K_1T = 90^\circ$. By definition 1 the Brocard point K_1 is on the circumcircle of $\triangle CDT \Rightarrow \sphericalangle CK_1T = \sphericalangle CDT = \delta_1$ (as inscribed angles). From the last two:

$$\sphericalangle P_3K_1C = \sphericalangle P_3K_1T + \sphericalangle CK_1T = 90^\circ + \delta_1. \tag{10}$$

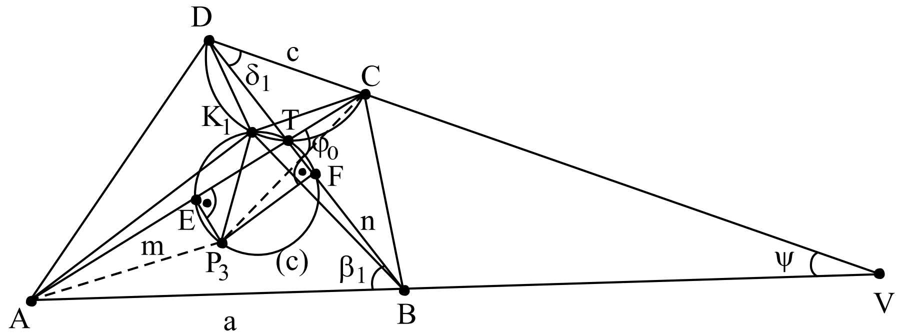


FIGURE 11. Theorem 4. Proof.

On the other hand $\triangle BAK_1 \sim \triangle DCK_1$ (from property 1) $\Rightarrow \sphericalangle BAK_1 = \sphericalangle DCK_1 \Leftrightarrow \sphericalangle VAK_1 = \sphericalangle DCK_1$. Therefore the quadrilateral is inscribed $\Rightarrow \sphericalangle AK_1C = 180^\circ - \sphericalangle AVC = 180^\circ - \psi$. From here and (10) we get:

$$\sphericalangle AK_1P_3 = \sphericalangle AK_1C - \sphericalangle P_3K_1C = \left(180^\circ - \psi\right) - \left(90^\circ + \delta_1\right) = 90^\circ - \left(\psi + \delta_1\right)$$

From the ΔBDV, where $\sphericalangle ABD = \beta_1$ is an external angle, $\sphericalangle BDV = \delta_1$ and $\sphericalangle BVD = \psi$ we get $\psi + \delta_1 = \beta_1$

$$\Rightarrow \ \sphericalangle AK_1P_3 = 90^\circ - \beta_1. \tag{11}$$

From the $\triangle P_3K_1C$ and $\triangle P_3K_1A$, by the cosine theorem, we obtain the equations, respectively:

$$P_3C^2 = K_1P_3^2 + K_1C^2 - 2K_1P_3.K_1C.\cos\sphericalangle P_3K_1C,$$

$$P_3A^2 = K_1P_3^2 + K_1A^2 - 2K_1P_3.K_1A.\cos\sphericalangle AK_1P_3.$$

If $\left|AC\right| = m,\ \left|BD\right| = n$, from $K_1C = \dfrac{mc}{2l_1},\ K_1A = \dfrac{ma}{2l_1}$ (property 3), the last two equations and (10) , (11):

$$P_3C^2 = K_1P_3^2 + \frac{m^2c^2}{4l_1^2} + 2K_1P_3\frac{mc}{2l_1}\cdot\sin\delta_1,$$

$$P_3A^2 = K_1P_3^2 + \frac{m^2a^2}{4l_1^2} - 2K_1P_3\frac{ma}{2l_1}\cdot\sin\beta_1.$$

We subtract the second of the last two equations from the first one, and because $P_3A = P_3C$, we get:

$$\frac{m^2a^2}{4l_1^2} - \frac{m^2c^2}{4l_1^2} = \frac{2K_1P_3.m}{2l_1}\cdot\left(c.\sin\delta_1 + a.\sin\beta_1\right)\cdot \tag{12}$$

On the other hand, we have successively (Fig. 11):

$$c.\sin\delta_1 + a.\sin\beta_1 = \frac{2}{n}\cdot\left(\frac{cn}{2}\cdot\sin\delta_1 + \frac{an}{2}\cdot\sin\beta_1\right) = \frac{2}{n}\cdot\left(S_{BCD} + S_{ABD}\right) =$$

$$= \frac{2}{n}\cdot S_{ABCD} = \frac{2}{n}\cdot\frac{mn\cdot\sin\varphi_0}{2} = m\cdot\sin\varphi_0.$$

Substituting with the resulting equality in (12), we arrive at the equation:

$$\frac{m^2\left(a^2-c^2\right)}{4l_1^{\,2}}=\frac{2K_1P_3.m}{2l_1}\cdot m.\sin\varphi_0$$

This result leads to the first of the proven equations (9). The second one is similarly proved.

Note. The next formulas are derived in a similar way:

$$K_1P_1=\frac{\left|m^2-n^2\right|}{4l_1.\sin\psi},\ K_2P_2=\frac{\left|m^2-n^2\right|}{4l_2.\sin\varphi}.$$

Final words. The intersection points of the bisectors of the opposite sides and bisectors of the diagonals of any quadrilateral have a number of other interesting properties as well. They are closely related to a pair of remarkable points in a quadrilateral called Antibrocarians and to a generalization of the circumcenter of an inscribed quadrilateral, called its second pseudocenter. We will discuss this at length in a separate article.

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