Two Mappings of a Tetrad of Points and the Transformations of Noticeable Points Generated by Them

Veselin N. Nenkov [1,a)], Sava Iv. Grozdev [2,b)], Asen P. Velchev [3,c)], Rayna M. Alashka [4,d)], Stanislav T. Stefanov [5,e)] and Haim M. Haimov [6]

[1] *Nikola Vaptsarov Naval Academy, Department of Mathematics and Physics, Faculty of Engineering, str. „Vasil Drumev" № 73, Varna 9002, Bulgaria*
[2] *University of Plovdiv "St. Paisii Hilendarski", Faculty of Mathematics and Informatics, "Bulgaria" blvd. № 236, Plovdiv 4027, Bulgaria*
[3] *Technical University of Sofia, Faculty of Applied Mathematics and Informatics, "Kliment Ohridski" blvd. №8, Sofia 1000, Bulgaria*
[4] *Sofia University "St. Kliment Ohridski", Faculty of Mathematics and Informatics, "James Bourchier" blvd. № 5, Sofia 1164, Bulgaria*
[5] *University of Architecture, Civil Engineering and Geodesy, Faculty of Transport Engineering, "Hristo Smirnenski" blvd. № 1, Sofia 1164, Bulgaria*
[6] *16 Bratia Shkorpil St., Varna 9000, Bulgaria*

[a)] vnenkov@mail.bg
[b)] sava.grozdev@gmail.com
[c)] Corresponding author: asen_v@abv.bg
[d)] ralashka@fmi.uni-sofia.bg
[e)] stanislav.toshkov@abv.bg

Abstract. We study a *set* of transformations of remarkable points of a triangle. *It* is generated by a mapping of four points in the plane of a given $\triangle ABC$ on the set of the triangles in that plane. The image of any tetrad of points by this mapping is a triangle perspective to $\triangle ABC$. We also consider the transformation of the four points onto a corresponding perspector. It generates various transformations of notable points, most of which are compositions of known mappings.

INTRODUCTION

Let A, B and C be non-collinear points, P – arbitrary Ceva point of $\triangle ABC$ and the points A_1, B_1, C_1 – functionally dependent from A, B, C. As the latter functions may be various, and Ceva's points – infinitely many for each triangle, there are infinitely many possible transformations (σ) of the points A, B, C onto P, A_1, B_1, C_1. We know various transformations of notable points in the plane (ABC): Euler transformation, Kosnita and Lalesco products (Grozdev, Dekov 2014 a), b), 2015), etc. Here we consider two mappings of four points in the plane of a triangle, which generate a set of products and transformations of remarkable points. For the tetrad of points (P, A_1, B_1, C_1), for which these mappings are defined, the image of the four points via the first mapping is a triangle $\triangle Q_aQ_bQ_c$ (we call it Q-triangle corresponding to the tetrad of points) perspective with $\triangle ABC$. The image of the tetrad (P, A_1, B_1, C_1) via the second mapping (we call it Q-mapping of the four points) is a point (we call it Q-point corresponding to these four), perspective of $\triangle Q_aQ_bQ_c$ and $\triangle ABC$ (we call the Q-point their perspector). Via the Q-mapping we define a product of two remarkable points in the plane of a given $\triangle ABC$, and transformations of notable points. The images of some of the latter via these transformations are also remarkable (for us) points, *not studied yet*. This can be easily explored by a computer program like "Discoverer" (Grozdev, Dekov 2013). Most of the obtained transformations are compositions of known mappings. The *images* of many notable points via some of them are points, collinear with *other* remarkable points. As (σ) belongs to an *uncountable infinite* class and each Q-mapping – too, the exploration of the various superpositions of (σ) and a Q-mapping is *uncountable infinite* too. As one may guess, we consider here only a few particular Q-mappings.

1) *Q*-TRIANGLE AND *Q*-POINT OF A TETRAD OF POINTS IN THE PLANE OF A GIVEN $\triangle ABC$

As we said, P from the tetrad (P, A_1, B_1, C_1) for which the Q-triangle and the Q-point are defined, is a Ceva's point for ΔABC. Let the Ceva's triangle for P be $A_0B_0C_0$ (Fig. 1). Construction of the Q-triangle for the tetrad (P, A_1, B_1, C_1) is as follows (A_1, B_1 and C_1 and related objects are missing in Fig. 1, for simplicity of the sketch):

1) through A_1, B_1, C_1 respective lines l_a, l_b and l_c, parallel to B_0C_0, C_0A_0 and A_0B_0;

2) $l_a \cap AC = Q_{ba}$, $l_a \cap AB = Q_{ca}$, $l_b \cap BC = Q_{ab}$, $l_b \cap AB = Q_{cb}$, $l_c \cap AC = Q_{bc}$, $l_c \cap BC = Q_{ac}$.

3) The points A, C and Q_{ab} form a circle k_{ab}, B, C and Q_{ba} form a circle k_{ba} (if $Q_{ba} = C$, then k_{ab} passes through A and C, and touches the line BC; if $Q_{ba} = C$ we construct k_{ba} the same way).

4) These two circles meet at the point C. Let their second common point be Q_c. We construct the points Q_a and Q_b in similar ways, as Q_a, Q_b and Q_c are vertices of the Q-triangle corresponding to the tetrad (P, A_1, B_1, C_1).

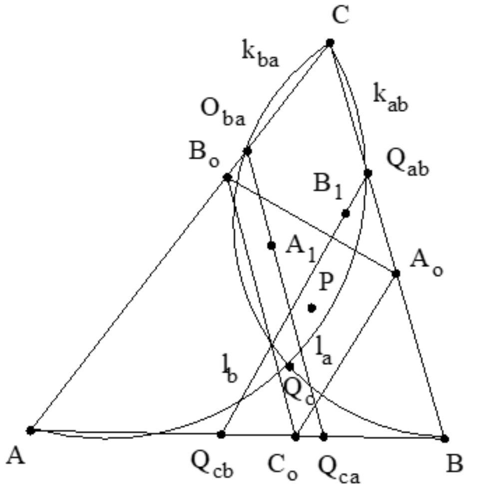

FIGURE 1. Ceva's triangle for the point P.

Let (σ) be such mapping that $A_1(1-y_a-z_a;\ y_a;\ z_a)$, $B_1(x_b;\ 1-x_b-z_b;\ z_b)$, $C_1(x_c;\ y_c;\ 1-x_c-y_c)$ in normalized barycentric coordinates. The barycentric coordinates of the vertices Q_a, Q_b and Q_c of the Q-triangle for the tetrad (P, A_1, B_1, C_1) are expressed by the coordinates of P, A_1, B_1 and C_1 in this way:

Theorem 1. If in barycentric coordinates $P(x : y : z)$, $A_1(1-y_a-z_a;\ y_a;\ z_a)$, $B_1(x_b;\ 1-x_b-z_b;\ z_b)$ and $C_1(x_c;\ y_c;\ 1-x_c-y_c)$, then the coordinates of the vertices of the Q-triangle of the tetrad (P, A_1, B_1, C_1) are:

$$Q_a\left(\frac{a_1c^2+a_2b^2-a^2}{a_1^{-1}+a_2^{-1}-1} : a_2b^2 : a_1c^2\right), Q_b\left(b_1a^2 : \frac{b_1a^2+b_2c^2-b^2}{b_1^{-1}+b_2^{-1}-1} : b_2c^2\right),$$

$$Q_c\left(c_2a^2 : c_1b^2 : \frac{c_1b^2+c_2a^2-c^2}{c_1^{-1}+c_2^{-1}-1}\right) \qquad (1)$$

<u>where:</u> $a = |BC|,\quad b = |CA|,\quad c = |AB|,$

$$\bar{b} = z(x+y)x_b + x(y+z)z_b,\ \bar{c} = y(x+z)x_c + x(y+z)y_c,\ \bar{a} = z(x+y)y_a + y(x+z)z_a, \qquad (2)$$

$$a_1 = \frac{\bar{b}}{z(x+y)};\ a_2 = \frac{\bar{c}}{y(x+z)};\ b_1 = \frac{\bar{c}}{x(y+z)};\ b_2 = \frac{\bar{a}}{z(x+y)};\ c_1 = \frac{\bar{a}}{y(x+z)};\ c_2 = \frac{\bar{b}}{x(y+z)}$$

Using Theorem 1, we can determine the barycentric coordinates of the vertices of the Q-triangles for any four remarkable points. Just put in its formulas the coordinates of the corresponding remarkable points, like this:

Example 1.1. Let G be the centroid of $\triangle ABC$ and $|AB| = c,\ |BC| = a,\ |CA| = b$. The barycentric coordinates of the vertices of the Q-triangle corresponding to the *four points* (G, G, G, G) are:

$$Q_a\left(2b^2+2c^2-3a^2 : 4b^2 : 4c^2\right);\ Q_b\left(4a^2 : 2a^2+2c^2-3b^2 : 4c^2\right);\ Q_c\left(4a^2 : 4b^2 : 2a^2+2b^2-3c^2\right).$$

Example 1.2. Let Ge be the Gergonne point of $\triangle ABC$, and J – its incenter. The coordinates of the vertices of the Q-triangle corresponding to the tetrad (G_e, J, J, J) are:

$$Q_a\left\{\left[2a\left(b^2+c^2\right)-a^2\left(a+b+c\right)\right]:2b^2\left(b+c\right):2c^2\left(b+c\right)\right\};$$
$$Q_b\left\{2a^2\left(a+c\right):\left[2b\left(c^2+a^2\right)-b^2\left(a+b+c\right)\right]:2c^2\left(a+c\right)\right\};$$
$$Q_c\left\{2a^2\left(a+b\right):2b^2\left(a+b\right):\left[2c\left(a^2+b^2\right)-c^2\left(a+b+c\right)\right]\right\}.$$

Example 1.3. Let G be the centroid of $\triangle ABC$, and $A_1B_1C_1$ be the Euler's triangle for G (Grozdev, Dekov 2014 a)). The coordinates of the vertices of the Q-triangle corresponding to the tetrad (G, A_1, B_1, C_1) are: $Q_a\left(b^2+c^2-3a^2:5b^2:5c^2\right)$; $Q_b\left(5a^2:a^2+c^2-3b^2:5c^2\right)$; $Q_c\left(5a^2:5b^2:a^2+b^2-3c^2\right)$.

It follows from the above formulas for the vertices of the Q-triangle corresponding to the tetrad of the above examples that these vertices lie on the corresponding symmedians of $\triangle ABC$. Interestingly, the Q-triangles to certain four points in the plane of a given $\triangle ABC$ are remarkable triangles of it.

Theorem 2. Let G be the centroid of the $\triangle ABC$ and A_1, B_1, C_1 – points on the lines BC, CA and AB. The Q-triangle for the tetrad (G, A_1, B_1, C_1) is the second Brocard 's triangle of $\triangle ABC$.

Proof: First find the vertex Q_C of the Q-triangle for the tetrad (G, A_1, B_1, C_1). According to the construction at the beginning, we construct the lines l_a and l_b through A_1 and B_1, parallel to B_0C_0 and C_0A_0 of the medial $\triangle A_0B_0C_0$ (Fig. 2). These lines contain the sides BC and CA, and hence the points Q_{ba} and Q_{ab}, where they cross AC and BC, coincide with C. We construct next the circles k_{ab} ; k_{ba}, , which pass through A, C ; B, C, and touch BC ; AC.

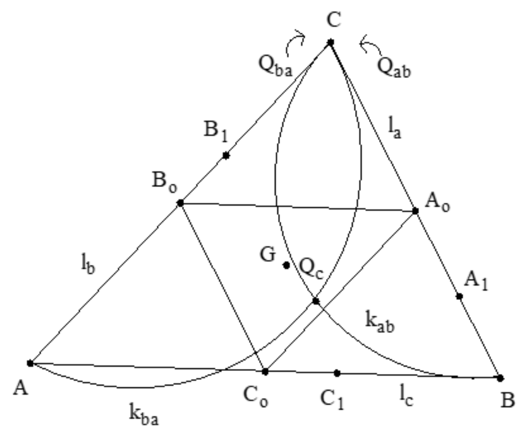

FIGURE 2. Theorem 2 Proof.

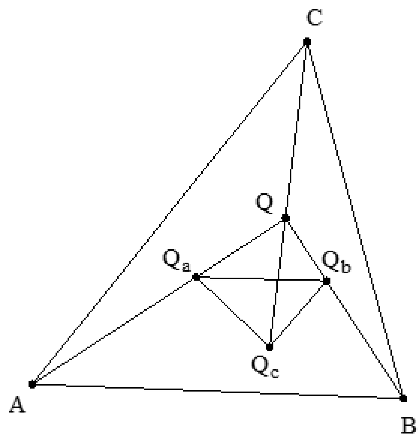

FIGURE 3. Theorem 4 Proof.

Their second common point Q_C is a vertex of the Q-triangle of the considered tetrad. This is exactly the vertex of the corresponding second Brocard's triangle (Grozdev, Nenkov 2012). The same way we prove the other two vertices of the Q-triangle for the tetrad (G, A_1, B_1, C_1) are vertices of the *second Brocard's triangle*. Therefore, *the last one* and the Q-triangle coincide. We prove analogically the following:

Theorem 3. Let $A_0B_0C_0$ be the Ceva's triangle for the point P, related to $\triangle ABC$, and A_1, B_1, and C_1 be points on the lines B_0C_0, C_0A_0 and A_0B_0 respectively. The Q-triangle corresponding to the tetrad (P, A_1, B_1, C_1) is the H-triangle of the point P (Haimov 2015).

Theorem 4. For every four points (P, A_1, B_1, C_1) their Q-triangle $\Delta Q_aQ_bQ_c$ is perspective to $\triangle ABC$.

Proof: With the notation of Theorem 1 we get the following equations of the lines AQ_a, BQ_b, CQ_c (Fig. 3):

$$-a_1c^2y+a_2b^2z=0$$
$$b_2c^2x-b_1a^2z=0$$
$$-c_1b^2x+c_2a^2y=0$$

It is easy to verify by Equations (2) that the determinant of the resulting system is zero. Therefore, the three lines intersect at a single point – the Q-perspector of $\triangle ABC$ and $\triangle Q_aQ_bQ_c$ (Paskalev, Chobanov – 1985).

Expressing the coordinates of the intersection point of the lines AQ_a, BQ_b, and CQ_c, by Theorem 4, we obtain the following coordinates (see Theorem 5) of the Q-point corresponding to the tetrad (P, A_1, B_1, C_1):

Theorem 5. Let the barycentric coordinates of the point P be (x : y : z) and the normalized barycentric coordinates of $A_1(1-y_a-z_a,\ y_a,\ z_a)$, $B_1(x_b,\ 1-x_b-z_b,\ z_b)$ and $C_1(x_c,\ y_c,\ 1-x_c-y_c)$. Then the coordinates of the Q-point of the tetrad (P, A_1, B_1, C_1), i.e. the perspector Q of $\triangle Q_aQ_bQ_c$ and $\triangle ABC$, are:

$$Q\left\{\frac{a^2}{x(y+z)\left[z(x+y)y_a+y(x+z)z_a\right]}:\frac{b^2}{y(z+x)\left[x(y+z)z_b+z(x+y)x_b\right]}:\frac{c^2}{z(x+y)\left[y(z+x)x_c+x(y+z)y_c\right]}\right\}$$

where $a=|BC|,\ b=|CA|,\ c=|AB|$.

2) Q-PRODUCT OF TWO POINTS IN THE PLANE OF A GIVEN TRIANGLE

Let X and Y be points in the plane (ABC), where X is a Ceva's point of the $\triangle ABC$. The mapping Q (X, Y), which transforms X and Y into the Q-point of the tetrad (X, Y, Y, Y), we call Q-product of X and Y. By Theorem 5 we obtain the coordinates of the Q-product of X and Y in the plane (ABC), from the coordinates of X, Y this way:

Theorem 6. Let in barycentric coordinates $X(x : y : z)$ and in normalized barycentric coordinates $Y(x_1, y_1, z_1)$. Then the coordinates of the Q-product of X and Y are:

$$Q\left\{\frac{a^2}{x(y+z)\left[z(x+y)y_1+y(x+z)z_1\right]}:\frac{b^2}{y(z+x)\left[x(y+z)z_1+z(y+x)x_1\right]}:\frac{c^2}{z(x+y)\left[y(z+x)x_1+x(y+z)y_1\right]}\right\}$$

where: $a=|BC|,\quad b=|CA|,\quad c=|AB|$.

If in the Q-product Q (X, Y) the one of the two points in the plane of a given $\triangle ABC$, is chosen as a remarkable point, we get a transformation relative to the other point. Such transformations also occur if we take the second point to coincide with the first or to be its image via certain transformation. As we will see, obtained by this way transformations, are predominantly compositions of known transformations.

Let g be the isogonal conjugation in a given $\triangle ABC$, φ – the isotomic conjugation and h – the complement. Then:

Theorem 7. The Q-product Q (P, P) of the point P with P is the isogonally conjugate point of the complement to the isotomically conjugate point of the complement to the point P, i.e. $Q(P, P) = gh\varphi h(P)$.

Proof: From Theorem 6 $\Rightarrow$ the barycentric coordinates of the product $Q(P, P)$ can be obtained by those of P:

$$Q\left[a^2\left(\frac{1}{x+z}+\frac{1}{x+y}\right)^{-1}:b^2\left(\frac{1}{x+y}+\frac{1}{y+z}\right)^{-1}:c^2\left(\frac{1}{y+z}+\frac{1}{x+z}\right)^{-1}\right] \qquad (3)$$

On the other hand, the complement $h(P)$ of the point P has coordinates: $(y+z:x+z:x+y)$, the isotomically conjugate point $\varphi[h(P)]$ of $h(P)$ has coordinates : $\left(\frac{1}{y+z}:\frac{1}{x+z}:\frac{1}{x+y}\right)$, and the complement to it $h\{\varphi[h(P)]\}$ – coordinates $\left(\frac{1}{x+z}+\frac{1}{x+y}:\frac{1}{x+y}+\frac{1}{y+z}:\frac{1}{y+z}+\frac{1}{x+z}\right)$.

Hence we obtain by (3) the coordinates of the isogonally conjugate point $g\left[h\varphi h(P)\right]$ of the point $h\varphi h(P)$.

Theorem 8. The transformations Q (P, P) and $Q[\varphi(P), G]$, where G is the centroid of a $\triangle ABC$, coincide.

Proof: As we saw above, the coordinates of the point $Q(P, P)$ are got by the formulas (3). Clearly, one may also obtain the coordinates of the point $Q[\varphi(P), G]$ by (3). Therefore, the two transformations coincide.

Theorem 9. The Q-product of a point P with the complement of P is equal to the isogonally conjugate point of the complement of the isotomically conjugate point of P, i.e.: $Q[P, h(P)] = gh\varphi(P)$.

Proof: By Theorem 6 we obtain the barycentric coordinates of the product $Q[P, h(P)]$ by those of P:

$$Q\left[a^2\left(\frac{1}{y}+\frac{1}{z}\right)^{-1}:b^2\left(\frac{1}{z}+\frac{1}{x}\right)^{-1}:c^2\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}\right]$$

It is easy to check, that the point $gh\varphi(P)$ has the same coordinates.

Corollary 9.1. If N is the Nagel point of a $\triangle ABC$, J – its incenter, Ge – its Gergonne point, then $Q(N,J)=gh(Ge)$.

Proof: We have $J=h(N)$ and $\varphi(N)=Ge$ (Paskalev, Chobanov – 1985). Then by Theorem 9, we obtain $Q(N,J)=Q(N,h(N))=gh\varphi(N)=gh(Ge)$.

The product of two points in the plane of a triangle, discussed above, gives another interesting transformation of remarkable points:

Theorem 10. The *Q*-product of the point *P* with the complement to the isotomically conjugate point *P* is the Lemoine point *L*, i.e. $Q\left(P,h\varphi\left(P\right)\right)=L$.

Proof: By Theorem 6 we obtain that the barycentric coordinates of the product $Q\left(P,h\varphi\left(P\right)\right)$ are:

$$Q\left\{\frac{a^2}{x\left(y+z\right)\left[z\left(x+y\right)\left(\frac{1}{x}+\frac{1}{z}\right)+y\left(x+z\right)\left(\frac{1}{x}+\frac{1}{y}\right)\right]}:\frac{b^2}{y\left(z+x\right)\left[x\left(y+z\right)\left(\frac{1}{x}+\frac{1}{y}\right)+z\left(x+y\right)\left(\frac{1}{y}+\frac{1}{z}\right)\right]}:\right.$$

$$\left.:\frac{c^2}{z\left(x+y\right)\left[y\left(z+x\right)\left(\frac{1}{y}+\frac{1}{z}\right)+x\left(z+y\right)\left(\frac{1}{x}+\frac{1}{z}\right)\right]}\right\}$$

or after simplification:

$$Q(a^2:b^2:c^2)$$

But these are exactly the coordinates of the point *L* of Lemoine.

3) Q-TRANSFORMATIONS OF REMARKABLE POINTS

The *Q*-transformations of remarkable points in the triangle are obtained as in the *Q*-product of two points, one is taken as a specific remarkable point. We will consider some examples:

Example 3.1. The centroid *G* of ΔABC is taken as the first point *X* in the *Q*-product *Q*(*X*,*Y*). The transformation *Q*(*G*,*Y*) is obtained. Using Theorem 6, we find the barycentric coordinates of the image of the point *P* after the *Q*(*G*,*Y*) transformation. We get the following:

Theorem 11. If the barycentric coordinates of the point *P* are (*x*: *y*: *z*), then the coordinates of its image *Q*(*G*,*P*) after the considered transformation will be:

$$Q\left(\frac{a^2}{y+z}:\frac{b^2}{z+x}:\frac{c^2}{x+y}\right),\text{ where: } a=\left|BC\right|,\ b=\left|CA\right|,\ c=\left|AB\right|.$$

The coordinates of the image of an arbitrary point *P* after the transformation *Q*(*G*,*Y*) coincide with those of the *H*-point of *P* (Haimov 2015). It follows that the transformation *Q*(*G*,*Y*)coincides with the *H*-mapping of the point *Y*. Thus, the following interesting consequences of Theory 11 are obtained:

Corollary 11.1. *Q*(*G*,*Y*) is product of complement mapping and isogonal conjugation, i,e, *Q*(*G*,*Y*) = *gh*(*Y*).

Corollary 11.2. The images of the orthocenter *H*, the centroid *G*, and the Nagel point *N* after the transformation *Q*(*G*, *Y*) are the orthocenter *H*, the Lemoine point *L* and the incenter *J*, respectively.

Via Theorem 11 we can easily find the coordinates of the images of different remarkable points after the transformation *Q*(*G*, *Y*). We obtain the following consequences of Theorem 11, which we will use later:

Corollary 11.3. The image of the incenter *J* after *Q*(*G*, *Y*) has coordinates: $Q\left(\frac{a^2}{b+c}:\frac{b^2}{c+a}:\frac{c^2}{a+b}\right)$.

Corollary 11.4. The image of the center of the *antibisectors* after *Q*(*G*, *Y*) is $Q\left(\frac{a}{b+c}:\frac{b}{c+a}:\frac{c}{a+b}\right)$.

Corollary 11.5. The image of Lemoine point after *Q*(*G*, *Y*) is $Q\left(\frac{a^2}{b^2+c^2}:\frac{b^2}{c^2+a^2}:\frac{c^2}{a^2+b^2}\right)$.

It turns out that the images of some remarkable points after *Q*(*G*, *Y*) are collinear with other remarkable points.

Theorem 12. The incenter *J*, its image via *Q*(*G*, *Y*) and the image of the center of the antibisectors via *Q*(*G*, *Y*) are collinear.

Proof: It is easy to verify this:

$$\begin{vmatrix} a & b & c \\ \dfrac{a^2}{b+c} & \dfrac{b^2}{c+a} & \dfrac{c^2}{a+b} \\ \dfrac{a}{b+c} & \dfrac{b}{c+a} & \dfrac{c}{a+b} \end{vmatrix} = 0$$

Here the row (*a*, *b*, *c*) are the coordinates of the point *J*, the row $\left(\dfrac{a^2}{b+c};\dfrac{b^2}{c+a};\dfrac{c^2}{a+b}\right)$ – *the image* of *J* after the transformation *Q*(*G*, *Y*) (see Corollary 11.3), and the row $\left(\dfrac{a}{b+c};\dfrac{b}{c+a};\dfrac{c}{a+b}\right)$ – the coordinates of the image after *Q*(*G*, *Y*) of the antibisectors' center (Corollary 11.4). Therefore, these notable points are collinear (Paskalev, Chobanov 1985).

The following theorems are proved the same way:

Theorem 13. The centroid *G*, the image of the Lemoine point *L* after *Q*(*G*, *Y*) and the isotomically conjugate point of the complement of *L* are collinear.

Theorem 14. The centroid *G*, the image of the antibisectors center after *Q*(*G*, *Y*) and the isotomically conjugate point of the complement of the incenter *J* are collinear.

Example 3.2. Take the Gergonne point as the first point of the product *Q*(*G*, *Y*). The result is the transformation *Q*(*Ge*, *Y*). By Theorem 6 we find the coordinates of the image of an arbitrary point *P* after the last transformation.

Theorem 15. Let barycentric coordinates *P*(*x*: *y* : *z*). Then the coordinates of the image of *P* after *Q*(*Ge*, *Y*) are:

$$Q\left(\frac{a}{cy+bz}:\frac{b}{az+cx}:\frac{c}{bx+ay}\right),\ \text{where: } a=|BC|,\ b=|CA|,\ c=|AB|.$$

In that formula we enter coordinates of some remarkable points and obtain the following:

Corollary 15.1. The image of the incenter *J* after the transformation *Q*(*Ge*, *Y*) is the Lemoine point *L*.

Corollary 15.2. The image of the Lemoine point *L* after the transformation *Q*(*Ge*, *Y*) is the isotomically conjugate point of the complement of the point *J*.

Corollary 15.3. The image of the centroid *G* after the transformation *Q*(*Ge*, *Y*) is the isogonally conjugate point of the complement to the center of the angle bisectors.

Finally, we will consider the following:

Example 3.3. Let the second one in the *Q*-product of two points *X* and *Y* coincides with the point of Lemoine *L*. We obtain the transformation *Q*(*X*, *L*).

By Theorem 6, we find the coordinates of the image of any point after the transformation. We get the following:

Theorem 16. Let the barycentric coordinates of the point *P* be (*x* : *y* : *z*). The coordinates of the image of *P* after the transformation *Q*(*X*, *L*) will be:

$$Q\left[a^2\left(\frac{c^2}{\frac{1}{x}+\frac{1}{y}}+\frac{b^2}{\frac{1}{x}+\frac{1}{z}}\right)^{-1}:b^2\left(\frac{a^2}{\frac{1}{y}+\frac{1}{z}}+\frac{c^2}{\frac{1}{y}+\frac{1}{x}}\right)^{-1}:c^2\left(\frac{b^2}{\frac{1}{x}+\frac{1}{z}}+\frac{a^2}{\frac{1}{z}+\frac{1}{y}}\right)^{-1}\right],\ \text{where:}$$

$$a=|BC|,\quad b=|CA|,\quad c=|AB|.$$

From here we get the following:

Corollary 16.1. The image of the point *P* after the transformation *Q*(*X*,*L*) is the point $ghgh\varphi(P)$.

ACKNOWLEDGEMENTS

The authors would like to thank the Research and Development Sector at the Technical University of Sofia for the financial support.

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