# Configurations of Pedal Circles of an Arbitrary Point in the Plane of a Polygon (Continuation)

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**Abstract**: Here we consider the mutual disposition of the pedal circles corresponding to an arbitrary point in the plane of an inscribed hexagon, with respect to a part of the triangles formed by the hexagon's sides and diagonals. Previously we have defined the Simpson circle of an arbitrary pentagon (see [1]). Here we prove that the Simpson circles of the six pentagons, formed by connecting each five among the six vertices of the inscribed hexagon, intersect at one point. Along the way, we establish an interesting property of the Simpson circle of an inscribed pentagon.

## INTRODUCTION

In [1] we consider configurations of pedal circles of an arbitrary point in the plane of a convex *quadrilateral/ pentagon/* with respect to the triangles, having as vertices consecutive vertices of *it*. There we define the concepts pedal image of a point with respect to a quadrilateral, Simpson circle and Simpson pentagon of a point with respect to a pentagon. Here we show that the Simpson circles of any point on the plane of an inscribed hexagon, with respect to the pentagons formed by its successive vertices, meet at one point, using these definitions and statements, formulated and proven in [1] and [3]:

A pedal circle of a point, which does not lie on the circumcircle of a triangle, we call the circle passing through the orthogonal projections of the point on the lines, which contain the sides of the triangle.

A polygon formed by part of the vertices of another polygon, we called its peak polygon, and a polygon formed by consecutive vertices of the initial polygon – its peripheral polygon.

**Theorem 1.** The pedal circles ($c_1$), ($c_2$), ($c_3$) and ($c_4$) of any point $X$ in the plane of the quadrilateral $ABCD$, which does not lie on the same circle with any three of its vertices, with respect to its peak triangles $ABC$, $BCD$, $CDA$ and $DAB$ meet at one point $Y$ (**Fig. 1**). *This point Y we called the pedal image of X with respect to ABCD*.

**Theorem 2.** The pedal images $A_1$, $B_1$, $C_1$, $D_1$ and $E_1$ of an arbitrary point $X$ in the plane of a random pentagon $ABCDE$ with respect to its peripheral quadrilaterals $EBCD$, $ACDE$, $CEAB$, etc., are concyclic (**Fig. 2**).
We called the respective circle the Simpson's circle for the point $X$ with respect to the pentagon $ABCDE$, and the pentagon $A_1B_1C_1D_1E_1$ – Simpson's pentagon for $X$.

## NEW RESULTS

**Theorem 3.** *ABCDEF* is an inscribed hexagon, *X* – a point on its plane, **not** on its circumcircle. Simpson's circles $k_A$, $k_B$, $k_C$, $k_D$, $k_E$ and $k_F$ for *X*, relative to the hexagon's peripheral pentagons *BCDEF*, *CDEFA*, etc., intersect at one point *Y* (**Fig. 3**).

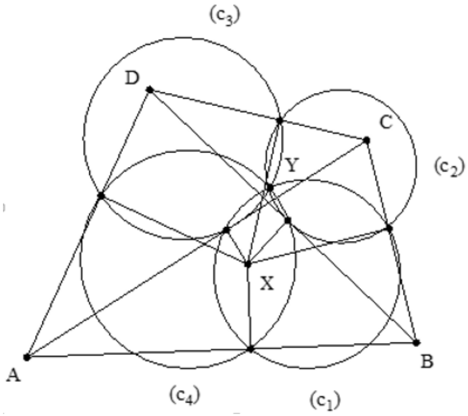


**FIGURE 1.** Shows **Theorem 1.**

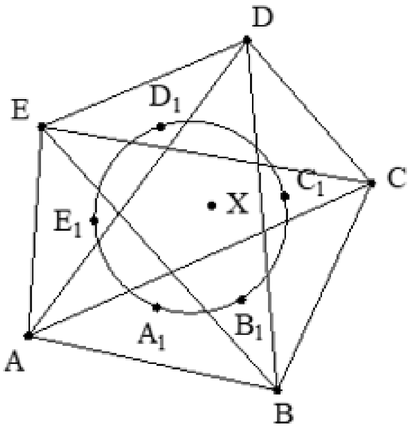


**FIGURE 2.** Shows **Theorem 2.**

Before proving this statement, we will introduce some notations and prove several lemmas:

Let *ABCDE* be a convex pentagon and *X* – a point in its plane (**Fig. 4**). Let $A_0$, $B_0$, $C_0$, $D_0$, $E_0$ be the orthogonal projections of *X* on the lines of the sides *AB*, *BC*, *CD*, *DE*, *EA* and those on the lines along the diagonals *EB*, *AC*, *BD*, *CE*, *DA* of the pentagon – $A_2$, $B_2$, $C_2$, $D_2$ and $E_2$. Let denote the pedal images of *X* relative to the peripheral quadrilaterals *BCDE*, *ACDE*, *ABDE, ABCE*, *ABCD* by $A_1$, $B_1$, $C_1$, $D_1$ and $E_1$ (**Fig. 2**), resp.

**Definition 1.** A vertex of the pentagon *ABCDE*, which does not belong to its peripheral quadrilateral, we call additional to the quadrilateral, and the quadrilateral – additional to the vertex (*D* is additional to *EABC*) (**Fig. 2**).

Using this definition, we will first define a correspondence between the peripheral triangles of a pentagon *ABCDE* and the peripheral triangles of its Simpson pentagon $A_1B_1C_1D_1E_1$ for an arbitrary point *X* in its plane. Take an arbitrary peripheral triangle of *ABCDE*, say $\triangle ABC$ (**Fig. 2**) and the three peripheral quadrilaterals *BCDE*, *CDEA*, *DEAB*, additional to the vertices *A*, *B*, *C*. The $\triangle A_1B_1C_1$ with vertices the pedal images $A_1$, $B_1$, $C_1$ of *X*, with respect to these peripheral quadrilaterals, we call conjugate to $\triangle ABC$ .

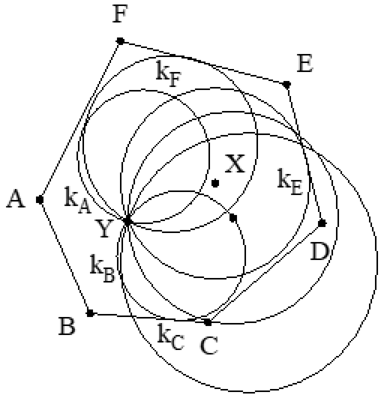


**FIGURE 3.** Shows **Theorem 3.**

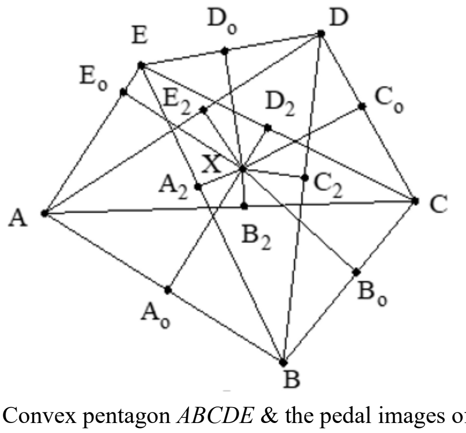


**FIGURE 4.** Convex pentagon *ABCDE* & the pedal images of point *X*.

In the proof of Theorem 3 we will use a property of Simpson's pentagon for an arbitrary point *X* in the plane of a circumscribed pentagon, but before formulating this property, we will prove several auxiliary statements:

**Lemma 1.** Let *ABC* be an arbitrary triangle and *X* be an arbitrary point, which lies: in the half-plane with contour the line *BC*, which does not contain the triangle, inside the angle *BAC* and not on the circumcircle of $\triangle ABC$ (**Fig. 5**). If $A_1$, $B_1$ and $C_1$ are the orthogonal projections of *X*, respectively on the lines *BC*, *CA* and *AB*, then for the angles of the pedal $\triangle A_1B_1C_1$ of the point *X* with respect to $\triangle ABC$ are satisfied these equations:

$$\begin{aligned} \sphericalangle A_1C_1B_1 &= \sphericalangle AXB - \sphericalangle ACB, \\ \sphericalangle A_1B_1C_1 &= \sphericalangle AXC - \sphericalangle ABC, \\ \sphericalangle B_1A_1C_1 &= 360^0 - \sphericalangle BXC - \sphericalangle BAC. \end{aligned} \tag{1}$$

**Proof 1.** Obviously $\sphericalangle A_1C_1B_1 = \sphericalangle XC_1B_1 - \sphericalangle XC_1A_1 \quad (\nabla)$ (Fig.5). From the inscribed quadrilateral $AC_1XB_1$ $\Rightarrow \sphericalangle XC_1B_1 = \sphericalangle XAB_1 \quad (*)$ and $\sphericalangle XC_1A_1 = \sphericalangle XBA_1 \quad (**)$ (from the inscribed quadrilateral $A_1C_1BX$ ).

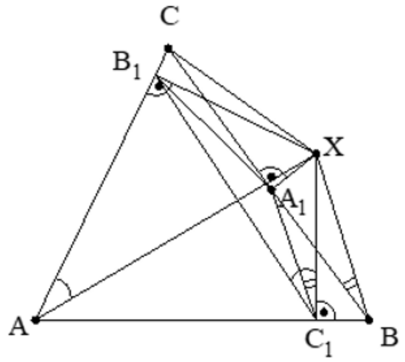


**FIGURE 5.** Shows **Lemma 1.**

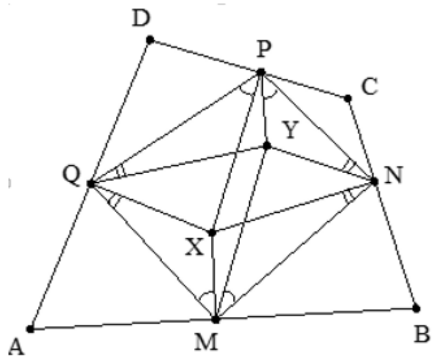


**FIGURE 6.** Shows **Lemma 2.**

Substituting the left sides of $(*), (**)$ with their right sides in $(\nabla)$, we get:

$$\sphericalangle A_1C_1B_1 = \sphericalangle XC_1B_1 - \sphericalangle XC_1A_1 = \sphericalangle XAB_1 - \sphericalangle XBA_1 = \sphericalangle XAC - \sphericalangle XBC,$$

i.e.:

$$\sphericalangle A_1C_1B_1 = \sphericalangle CAX - \sphericalangle CBX \tag{2}$$

On the other hand:

$$\sphericalangle AXB = 180^0 - \sphericalangle BAX - \sphericalangle ABX = 180^0 - (\sphericalangle CAB - \sphericalangle CAX) - (\sphericalangle ABC + \sphericalangle CBX) =$$
$$= (180^0 - \sphericalangle CAB - \sphericalangle ABC) + \sphericalangle CAX - \sphericalangle CBX = \sphericalangle ACB + \sphericalangle CAX - \sphericalangle CBX,$$

i.e.:

$$\sphericalangle AXB = \sphericalangle ACB + \sphericalangle CAX - \sphericalangle CBX.$$

From the last one we get: $\sphericalangle AXB - \sphericalangle ACB = \sphericalangle CAX - \sphericalangle CBX$.

As the right-hand sides of the last equation and equation (2) are the same, by equating their's left sides, we get:

$$\sphericalangle A_1C_1B_1 = \sphericalangle AXB - \sphericalangle ACB.$$

Thus we proved the first of the equations (1). The other two can be proved analogously. Let us now recall this:

**Definition 2.** Two lines through the vertex of an arbitrary angle are called isogonal with respect to it, if they form equal angles with its angle bisector, and therefore with the arms of the angle.

**Lemma 2.** Let *X* be an arbitrary point in the plane of an inscribed quadrilateral *ABCD* (**Fig. 6**), not lying on its circumcircle and different from its Miquel's point. Further: let *M*, *N*, *P* and *Q* be *X*'s orthogonal projections on the lines *AB*, *BC*, *CD* and *DA* which contain the quadrilateral's respective sides. *X* and its pedal image *Y* with respect to the quadrilateral *ABCD* lie on isogonal lines with respect to each of the angles of the quadrilateral *MNPQ*.

This lemma is proven in [2] (see the corollary of Theorem 8 at the end of the paper).

**Lemma 3.** Let $A_1B_1C_1D_1E_1$ be the Simpson's pentagon for the point *X* with respect to an inscribed pentagon *ABCDE*. Let consider an arbitrary angle of the first one, say $\sphericalangle E_1A_1B_1$ (**Fig. 7**). It is the angle of the peripheral $\triangle E_1A_1B_1$ of the pentagon $A_1B_1C_1D_1E_1$. Let consider the peripheral $\triangle EAB$ of the pentagon *ABCDE*, conjugate to $\triangle E_1A_1B_1$ and the corresponding angle of the pedal $\triangle C_0A_2D_0$ of the point *X* relative to it – namely $\sphericalangle C_0A_2D_0$. The following equalities hold: $\sphericalangle E_1A_1B_1 = \sphericalangle C_0A_2D_0$, $\sphericalangle A_1B_1C_1 = \sphericalangle D_0B_2E_0$, $\sphericalangle B_1C_1D_1 = \sphericalangle A_0C_2E_0$, ...

**Proof 3.** The vertex $A_1$ of the Simpson's pentagon $A_1B_1C_1D_1E_1$ for the point *X* is the pedal image of *X* with respect to the quadrilateral *BCDE* (**Fig. 7**) and $B_1$ is the pedal image of *X* with respect to *CDEA*. $\triangle ECD$ is a peak triangle in both quadrilaterals, so the pedal images $A_1$ and $B_1$ of *X* relative to them, lie on the pedal circle *k* of the point *X* relative to this triangle. The circle *k* also contains the orthogonal projections of *X* with respect to the sides *CD* and *ED* of $\triangle ECD$ – the points $A_0$ and $B_0$. Therefore, the quadrilateral $A_1A_0B_0B_1$ is inscribed in *k*. Therefore, if *P* is an arbitrary point on the continuation of $A_0A_1$, then $\sphericalangle PA_1B_1 = \sphericalangle B_1B_0A_0$. Analogically, $\sphericalangle PA_1E_1 = \sphericalangle E_1E_0A_0$ (from the inscribed quadrilateral $A_1E_1E_0A_0$ ). From the last two equations we get:

$$\sphericalangle E_1A_1B_1 = \sphericalangle PA_1E_1 + \sphericalangle PA_1B_1 = \sphericalangle E_1E_0A_0 + \sphericalangle B_1B_0A_0$$

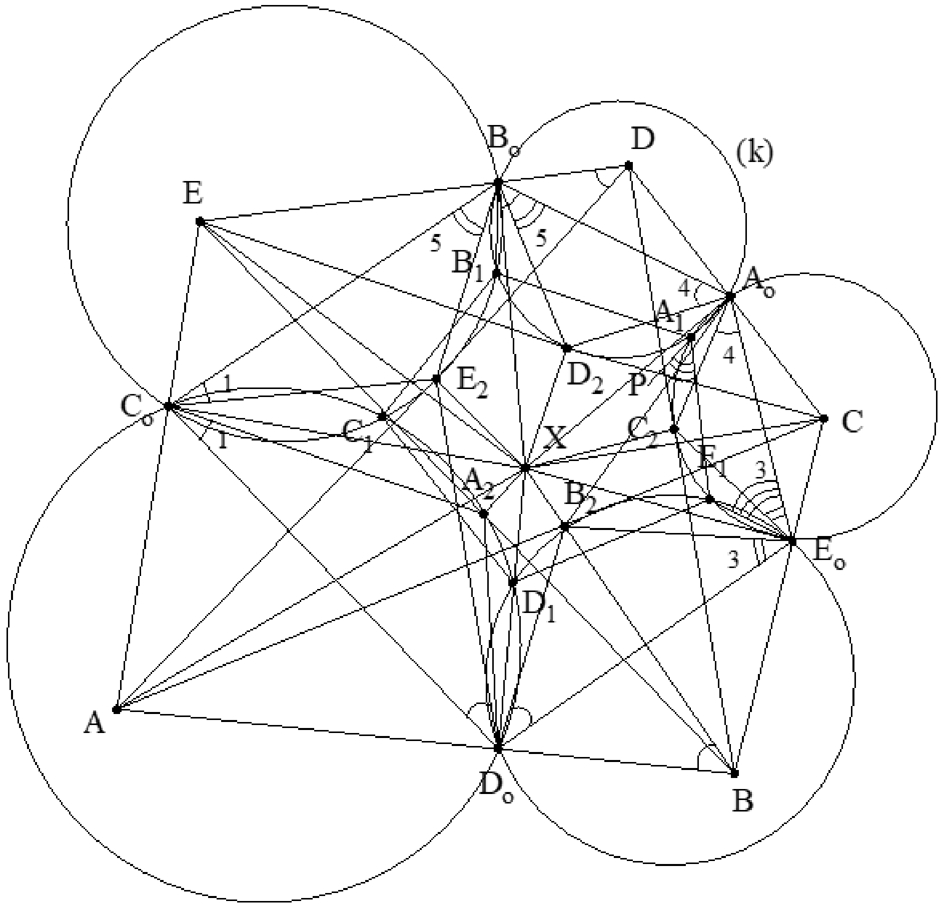


**FIGURE 7.** Shows **Lemmas 3, 4 and 5, and Theorem 4.**

The pedal image $B_1$ of $X$ with respect to the inscribed quadrilateral $CDEA$ <u>and</u> the point $X$ lie on lines, isogonal with respect to the $\sphericalangle A_0B_0C_0$ of the pedal quadrilateral $A_0B_0C_0B_2$ of $X$ with respect to $CDEA$ (according to Lemma 2). Therefore $\sphericalangle B_1B_0A_0 = \sphericalangle XB_0C_0$ (according to definition 2). Analogically, from the quadrilateral $A_0E_0D_0E_2$ $\Rightarrow \sphericalangle E_1E_0A_0 = \sphericalangle XE_0D_0$ . We substitute the last two equations in the above and get:

$$\sphericalangle E_1A_1B_1 = \sphericalangle E_1E_0A_0 + \sphericalangle B_1B_0A_0 = \sphericalangle XE_0D_0 + \sphericalangle XB_0C_0$$

On the other hand $\sphericalangle XE_0D_0 = \sphericalangle XBD_0$ (from the inscribed quadrilateral $XD_0BE_0$) and $\sphericalangle XB_0C_0 = \sphericalangle XEC_0$ (from the inscribed quadrilateral $XC_0EB_0$). We substitute these equations in the above and get:

$$\sphericalangle E_1A_1B_1 = \sphericalangle XE_0D_0 + \sphericalangle XB_0C_0 = \sphericalangle XBD_0 + \sphericalangle XEC_0 .$$

On the other hand, from the quadrilateral $ABXE$, we have:

$$\sphericalangle XBD_0 + \sphericalangle XEC_0 = \sphericalangle XBA + \sphericalangle XEA = 360^o - \sphericalangle EXB - \sphericalangle EAB.$$

Therefore:

$$\sphericalangle E_1A_1B_1 = \sphericalangle XBD_0 + \sphericalangle XEC_0 = 360^o - \sphericalangle EXB - \sphericalangle EAB.$$

At the same time, for the pedal $\triangle C_0A_2D_0$ of $X$ with respect to $\triangle EAB$ (according to Lemma 1):

$$\sphericalangle C_0A_2D_0 = 360^o - \sphericalangle EXB - \sphericalangle EAB .$$

Comparing the right sides of the last two equations we get the first equation proved: $\sphericalangle E_1A_1B_1 = \sphericalangle C_0A_2D_0$. The other equations can be analogically proved.

**Lemma 4.** Let $ABCDE$ be an inscribed pentagon and $X$ – a point *not* on its circumcircle. For the initially accepted notations, the following equations are satisfied (**Fig. 7**):

$$\sphericalangle A_2C_0D_0 = \sphericalangle B_0C_0E_2 = \sphericalangle 1,\ \sphericalangle B_2D_0E_0 = \sphericalangle A_2D_0C_0 = \sphericalangle 2,\ \sphericalangle C_2E_0A_0 = \sphericalangle B_2E_0D_0 = \sphericalangle 3 \quad (3)$$

and so on (i.e. the adjacent angles of the pedal triangles of point $X$ relative to the successive peripheral triangles of $ABCDE$ are equal to each other).

**Proof 4.** $A_2C_0D_0$ is the pedal triangle of $X$ with respect to $\triangle ABE$ (**Fig. 7**), so by Lemma 1 we have:

$$\sphericalangle A_2C_0D_0 = \sphericalangle AXE - \sphericalangle ABE$$

Analogically, from $\triangle ADE$ , again by Lemma 1, we have:

$$\sphericalangle B_0C_0E_2 = \sphericalangle AXE - \sphericalangle ADE$$

Since $\sphericalangle ABE = \sphericalangle ADE$ (as inscribed angles), comparing the right sides of the last two equations, we get $\sphericalangle A_2C_0D_0 = \sphericalangle B_0C_0E_2$ (proof of the first one of the chain of equations (3)). The other two we prove analogously.

**Lemma 5.** If *ABCDE* is an inscribed pentagon and *X* – a point in its plane, but **not** on its circumcircle (**Fig. 7**). For the initially accepted notations:

$$\sphericalangle A_0D_2B_0 + \sphericalangle E_0B_2D_0 = \sphericalangle E_2C_0B_0 + 180^o \qquad (4)$$

**Proof 5.** Since $\triangle A_0D_2B_0$ is the pedal triangle of *X* with respect to $\triangle ECD$, according to Lemma 1 (**Fig. 7**):

$$\sphericalangle A_0D_2B_0 = 360^o - \sphericalangle EXC - \sphericalangle EDC$$

From $\triangle ABC$ , also by Lemma 1, it follows the equality:

$$\sphericalangle E_0B_2D_0 = 360^o - \sphericalangle AXC - \sphericalangle ABC$$

We add the last two equalities and get:

$$\sphericalangle A_0D_2B_0 + \sphericalangle E_0B_2D_0 = 720^0 - \sphericalangle EXC - \sphericalangle EDC - \sphericalangle AXC - \sphericalangle ABC \qquad (5)$$

On the other hand, $\sphericalangle ADE = \sphericalangle EDC - \sphericalangle ADC$ (Fig. 7) and $\sphericalangle E_2C_0B_0 = \sphericalangle AXE - \sphericalangle ADE$ (according to Lemma 1). Using both of them, we get consequently:

$$\sphericalangle E_2C_0B_0 = \sphericalangle AXE - \sphericalangle ADE = 360^o - \sphericalangle AXC - \sphericalangle EXC - (\sphericalangle EDC - \sphericalangle ADC)$$

i.e.:

$$\sphericalangle E_2C_0B_0 = (360^o - \sphericalangle AXC - \sphericalangle EXC - \sphericalangle EDC) + \sphericalangle ADC =$$
$$= \left[720^o - (\sphericalangle AXC + \sphericalangle EXC + \sphericalangle EDC) - 360^o\right] + \left(180^o - \sphericalangle ABC\right)$$

hence:

$$\sphericalangle E_2C_0B_0 = \left(720^o - \sphericalangle AXC - \sphericalangle EXC - \sphericalangle EDC - \sphericalangle ABC\right) - 180^o .$$

Substitute the right part of the last equation with the right side of equation (5), we get:

$$\sphericalangle E_2C_0B_0 = \sphericalangle A_0D_2B_0 + \sphericalangle E_0B_2D_0 - 180^o$$

From this immediately follows the equality which we are proving.

**Lemma 6.** Any two inscribed pentagons with equal corresponding angles are similar.

**Proof 6.** Let *ABCDE* and $A_1B_1C_1D_1E_1$ be two inscribed pentagons of equal corresponding angles (Fig. $8^{a,b}$ ):

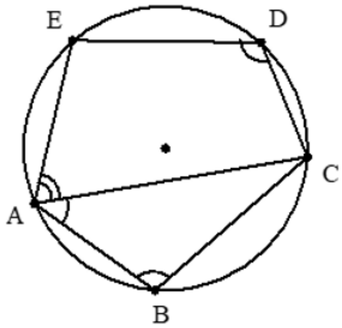


**FIGURE** $8^a$ **.** Shows **Lemma 6.**

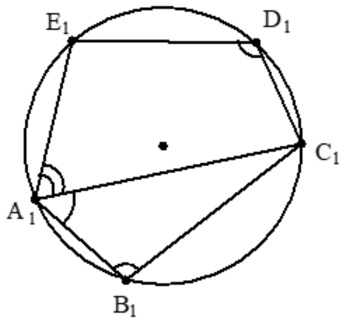


**FIGURE** $8^b$ **.** Shows **Lemma 6.**

To prove their similarity, let show first that $\triangle ABC \sim \triangle A_1B_1C_1$. It's given that $\sphericalangle ABC = \sphericalangle A_1B_1C_1$ . Further: $\sphericalangle EAC = 180^o - \sphericalangle EDC = 180^o - \sphericalangle E_1D_1C_1 = \sphericalangle E_1A_1C_1$ i.e. $\sphericalangle EAC = \sphericalangle E_1A_1C_1$. Therefore $\sphericalangle CAB = \sphericalangle EAB - \sphericalangle EAC = \sphericalangle E_1A_1B_1 - \sphericalangle E_1A_1C_1 = \sphericalangle C_1A_1B_1$ . We got two equal respective angles in $\triangle ABC$ and $\triangle A_1B_1C_1 \Rightarrow$ they are similar. The similarity of the other peripheral triangles of the pentagons may be analogically proven. Hence the pentagons are composed of equally placed similar triangles, so they are similar.

**Lemma 7.** Let *ABCDE* be an inscribed pentagon and *X* – a point in its plane and **not** on its circumcircle. There exists an inscribed pentagon $\overline{A}\overline{B}\overline{C}\overline{D}\overline{E}$ (**Fig. 9**), the successive peripheral triangles of which are correspondingly similar to the pedal triangles of *X* with respect to the consecutive peripheral triangles of *ABCDE*.

**Proof 7.** We use the notations accepted in the beginning. First we construct the vertices $\overline{C}$, $\overline{D}$ and $\overline{E}$ of the required pentagon $\overline{ABCDE}$ so that they coincide with the points $B_0$, $D_2$ and $A_0$ (Fig. 9). Let *k* be the circumcircle of $\triangle\overline{CDE}$. On *k* we choose a point $\overline{A}$ so that $\measuredangle\overline{ADE} = \measuredangle B_0C_oE_2 = \measuredangle 1$. Since $\measuredangle\overline{EAD} = \measuredangle\overline{ECD}$ (inscribed angles) and $\measuredangle\overline{ECD} = \measuredangle 5 = \measuredangle E_2B_0C_0$ (according to Lemma 4), then $\measuredangle\overline{EAD} = \measuredangle\overline{ECD} = \measuredangle E_2B_0C_0 = \measuredangle 5$. Hence $\triangle\overline{ADE}$ and $\triangle E_2B_0C_0$ have two equal angles, and therefore they are similar. Let $\overline{B}$ be a point on the circle *k* such that $\measuredangle\overline{BEA} = \measuredangle A_2D_0C_0 = \measuredangle 2$. As $\measuredangle\overline{EBA} = \measuredangle\overline{EDA}$ as inscribed angles, and $\measuredangle\overline{EDA} = \measuredangle 1 = \measuredangle D_0C_0A_2$, then $\measuredangle\overline{EBA} = \measuredangle D_0C_0A_2$. Hence $\triangle\overline{EBA}$ and $\triangle D_0C_0A_2$ have two equal angles, i.e. they are similar. To prove the lemma, it remains to prove that $\triangle\overline{ABC} \sim \triangle E_0B_2D_0$ and $\triangle CBD \sim \triangle E_0A_0C_2$.

As inscribed angles $\measuredangle\overline{BCA} = \measuredangle BEA$ and $\measuredangle\overline{BEA} = \measuredangle 2 = \measuredangle A_2D_0C_0$ (from the definition of $\overline{B}$) $\Rightarrow \measuredangle\overline{BCA} = \measuredangle A_2D_0C_0$. But $\measuredangle A_2D_0C_0 = \measuredangle E_0D_0B_2$ (by Lemma 4) $\Rightarrow \measuredangle\overline{BCA} = \measuredangle A_2D_0C_0 = \measuredangle E_0D_0B_2$. To prove that $\triangle\overline{ABC} \sim \triangle E_0B_2D_0$, we have to prove that $\measuredangle\overline{ABC} = \measuredangle E_0B_2D_0$. According to Lemma 5:

$$\measuredangle A_0D_2B_0 + \measuredangle E_0B_2D_0 = \measuredangle E_2C_0B_0 + 180^o$$

But $\measuredangle A_0D_2B_0 = 180^o - \measuredangle\overline{EBC}$ and $\measuredangle E_2C_0B_0 = \measuredangle 1 = \measuredangle\overline{ADE} = \measuredangle\overline{ABE}$.

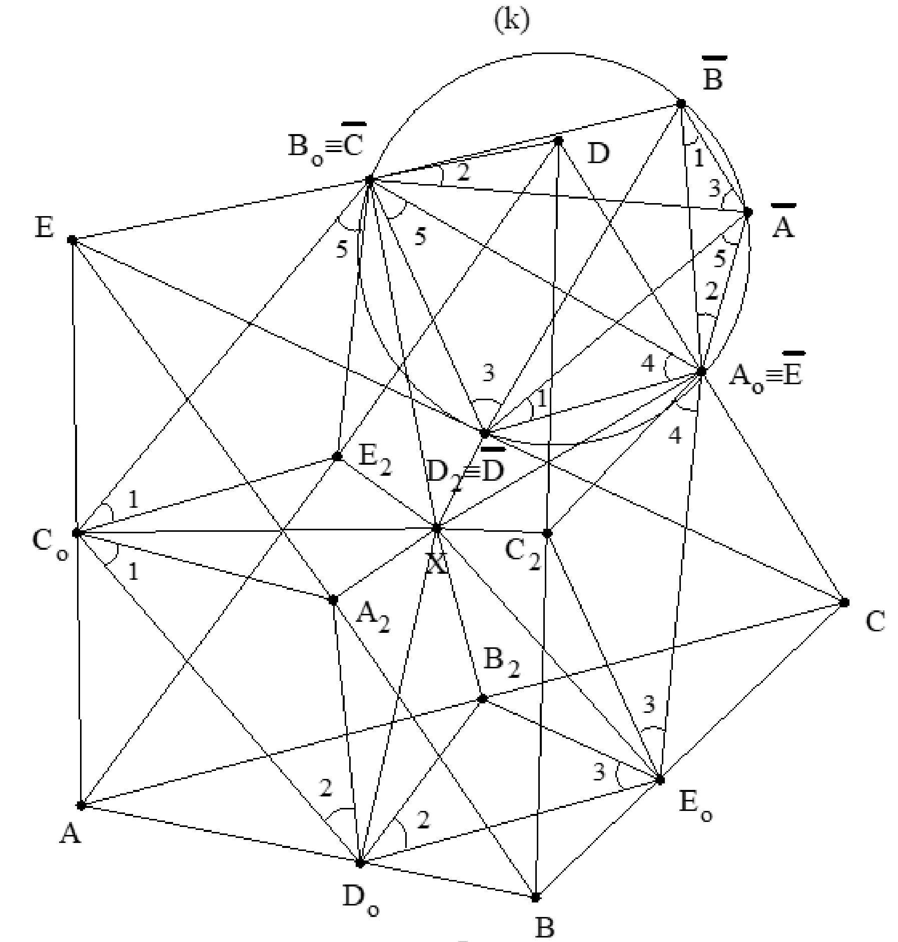


**FIGURE 9.** Shows **Lemma 7 and Theorem 4.**

We substitute these equations in the above and get:

$$(180^o - \measuredangle\overline{EBC}) + \measuredangle E_0B_2D_0 = \measuredangle\overline{ABE} + 180^o$$

i.e.:

$$\measuredangle\overline{ABE} + \measuredangle\overline{EBC} = \measuredangle E_0B_2D_0.$$

But $\sphericalangle\overline{ABE}+\sphericalangle\overline{EBC}=\sphericalangle\overline{ABC}$, hence we get the wanted $\sphericalangle\overline{ABC}=\sphericalangle E_0B_2D_0 \Rightarrow$ the similarity $\triangle\overline{ABC}\sim\triangle E_0B_2D_0$ is proven. It remains to prove that $\triangle\overline{CBD}\sim\triangle E_0A_0C_2$. From the proven $\triangle\overline{ABC}\sim\triangle E_0B_2D_0$ we have $\sphericalangle\overline{CAB}=\sphericalangle D_0E_0B_2$. But $\sphericalangle D_0E_0B_2=\sphericalangle A_0E_0C_2$ (from Lemma 4) $\Rightarrow\sphericalangle\overline{CAB}=\sphericalangle A_0E_0C_2$, so $\sphericalangle\overline{CDB}=\sphericalangle\overline{CAB}=\sphericalangle A_0E_0C_2$. Further: $\sphericalangle\overline{CBD}=\sphericalangle\overline{CED}=\sphericalangle 4=\sphericalangle E_0A_0C_2$. In $\triangle\overline{CDB}$ and $\triangle A_0E_0C_2$ we got two respectively equal angles $\Rightarrow$ they are similar. This proves the lemma.

We can now formulate and prove the property in question of the peripheral triangles of Simpson's pentagon for an arbitrary point *X* in the plane of an inscribed pentagon:

**Theorem 4.** Any peripheral triangle of the Simpson's pentagon $A_1B_1C_1D_1E_1$ for a point *X* which lies *not* on the circumcircle of the inscribed pentagon *ABCDE*, is similar to the pedal Δ of *X* with respect to its conjugate Δ, peripheral of *ABCDE* (in our notation $\triangle A_1B_1C_1\sim\triangle D_0B_2E_0$, $\triangle B_1C_1D_1\sim\triangle A_0C_2E_0$, $\triangle C_1D_1E_1\sim\triangle B_0D_2A_0$ (**Fig. 7**).

**Proof:** According to Lemma 7 there exists an inscribed pentagon $\overline{ABCDE}$, with peripheral triangles $\overline{ABC}$, $\overline{BCD}$, $\overline{CDE}$, etc. similar to the pedal triangles $D_0B_2E_0$, $A_0C_2E_0$, $B_0D_2A_0$, ... of *X* with respect to the peripheral triangles *ABC*, *BCD*, *CDE*, etc. of the given pentagon *ABCDE* (Fig. 9). Then the corresponding angles of $\triangle\overline{ABC}$, $\triangle\overline{BCD}$, $\triangle\overline{CDE}$, etc. and those of the pedal $\triangle D_0B_2E_0$, $\triangle A_0C_2E_0$, $\triangle B_0D_2A_0$, etc., are equal. Hence:

$$\sphericalangle\overline{ABC}=\sphericalangle D_0B_2E_0,\ \sphericalangle\overline{BCD}=\sphericalangle A_0C_2E_0,\ \sphericalangle\overline{CDE}=\sphericalangle B_0D_2A_0,\text{ etc.}$$

At the same time, for the Simpson's pentagon $A_1B_1C_1D_1E_1$ for the point *X* with respect to the pentagon *ABCDE*, according to Lemma 3, the following equations are satisfied (Fig. 7):

$$\sphericalangle A_1B_1C_1=\sphericalangle D_0B_2E_0,\quad \sphericalangle B_1C_1D_1=\sphericalangle A_0C_2E_0,\quad \sphericalangle C_1D_1E_1=\sphericalangle B_0D_2A_0,\text{ etc.}$$

It follows from them that:

$$\sphericalangle\overline{ABC}=\sphericalangle A_1B_1C_1,\quad \sphericalangle\overline{BCD}=\sphericalangle B_1C_1D_1,\quad \sphericalangle\overline{CDE}=\sphericalangle C_1D_1E_1,\text{ etc.}$$

We get that the inscribed pentagons $\overline{ABCDE}$ and $A_1B_1C_1D_1E_1$ have correspondingly equal angles, therefore they are similar (according to Lemma 6). Then their corresponding peripheral triangles are similar, i.e.:

$$\triangle\overline{ABC}\sim\triangle A_1B_1C_1,\quad \triangle\overline{BCD}\sim\triangle B_1C_1D_1,\quad \triangle\overline{CDE}\sim\triangle C_1D_1E_1,\text{ etc.}$$

But the peripheral triangles $\overline{ABC}$, $\overline{BCD}$, $\overline{CDE}$, etc. of the pentagon $\overline{ABCDE}$ are similar to the pedal triangles $D_0B_2E_0$, $A_0C_2E_0$, $B_0D_2A_0$ etc. of the point *X* with respect to the successive peripheral triangles of the pentagon *ABCDE* (according to Lemma 7), i.e.:

$$\triangle\overline{ABC}\sim\triangle D_0B_2E_0,\quad \triangle\overline{BCD}\sim\triangle A_0C_2E_0,\quad \triangle\overline{CDE}\sim\triangle B_0D_2A_0,\text{ etc.}$$

Then the similar (according to what we have just proved) peripheral triangles $A_1B_1C_1$, $B_1C_1D$, $C_1D_1E_1$ etc. of the Simpson's pentagon $A_1B_1C_1D_1E_1$ for the point *X* with respect to *ABCDE* will be similar to the pedal triangles $D_0B_2E_0$, $A_0C_2E_0$, $B_0D_2A_0$ etc. of the point *X* with respect to the successive peripheral triangles of the pentagon *ABCDE*, i.e. these similarities will hold:

$$\triangle A_1B_1C_1\sim\triangle D_0B_2E_0,\quad \triangle B_1C_1D_1\sim\triangle A_0C_2E_0,\quad \triangle C_1D_1E_1\sim\triangle B_0D_2A_0,\text{ etc.}$$

This proves Theorem 4. We can now prove Theorem 3 (see it on p.1 and Fig. 3).

**Proof of Theorem 3:** The Simpson's circle $k_B$ for the point *X* with respect to the pentagon *CDEFA*, passes through the pedal images $F_B$, $A_B$ и $C_B$ of the point *X* with respect to its peripheral quadrilaterals *ACDE*, *CDEF*, and *DEFA* (as defined by Simpson's circle) (Fig. 10). Similarly, Simpson's circle $k_F$ for the point *X* with respect to the pentagon *ABCDE* passes through the pedal images $A_F$, $B_F$ и $C_F$ of *X* with respect to its peripheral quadrilaterals *BCDE*, *CDEA*, and *DEAB*. We get that both the points $B_F$ and $F_B$ are pedal images of *X* with respect to the quadrilateral *ACDE*, therefore $B_F = F_B$. Simpson's circle $k_C$ for the point *X*, with respect to the pentagon *DEFAB*, passes through the pedal images $F_C$, $A_C$ and $B_C$ of *X* with respect to its peripheral quadrilaterals *ABDE*, *BDEF* and *DEFA*. We obtain that both the points $F_C$ and $C_F$ are pedal images of *X* with respect to the quadrilateral *ABDE* and that the points $B_C$ and $C_B$ are pedal images of *X* with respect to the quadrilateral *DEFA*. Therefore:

$$F_C = C_F \text{ and } B_C = C_B.$$

Let $A_0$, $B_0$, $F_0$, $A_2$, $B_2$ and $B_3$ be the orthogonal projections of $X$ on the lines $AB$, $BC$, $AF$, $BF$, $AC$ and $FC$ respectively. $C_BA_BF_B$ is the peripheral triangle of the Simpson's pentagon for $X$ with respect to the pentagon $ACDEF$, and $F_0B_3B_2$ is the pedal triangle of $X$ with respect to the conjugate peripheral triangle $CAF$ of $ACDEF$. Then (according to Theorem 4) $\Delta C_BA_BF_B \sim \Delta F_0B_3B_2$ and then $<C_BA_BF_B = <F_0B_3B_2$. On the other hand, for the pedal $\Delta F_0B_3B_2$ of $X$ with respect to $\Delta CAF$ by Lemma 1 angles: $< F_0B_3B_2 = < FXC - < FAC$. Therefore:

$$\sphericalangle C_BA_BF_B = \sphericalangle F_0B_3B_2 = \sphericalangle FXC - \sphericalangle FAC \tag{6}$$

Analogically:

$$\sphericalangle C_FA_FB_F = \sphericalangle A_0B_0B_2 = \sphericalangle BXC - \sphericalangle BAC \tag{7}$$

and

$$\sphericalangle F_CA_CB_C = \sphericalangle A_0A_2F_0 = 360^o - \sphericalangle BXF - \sphericalangle BAF \tag{8}$$

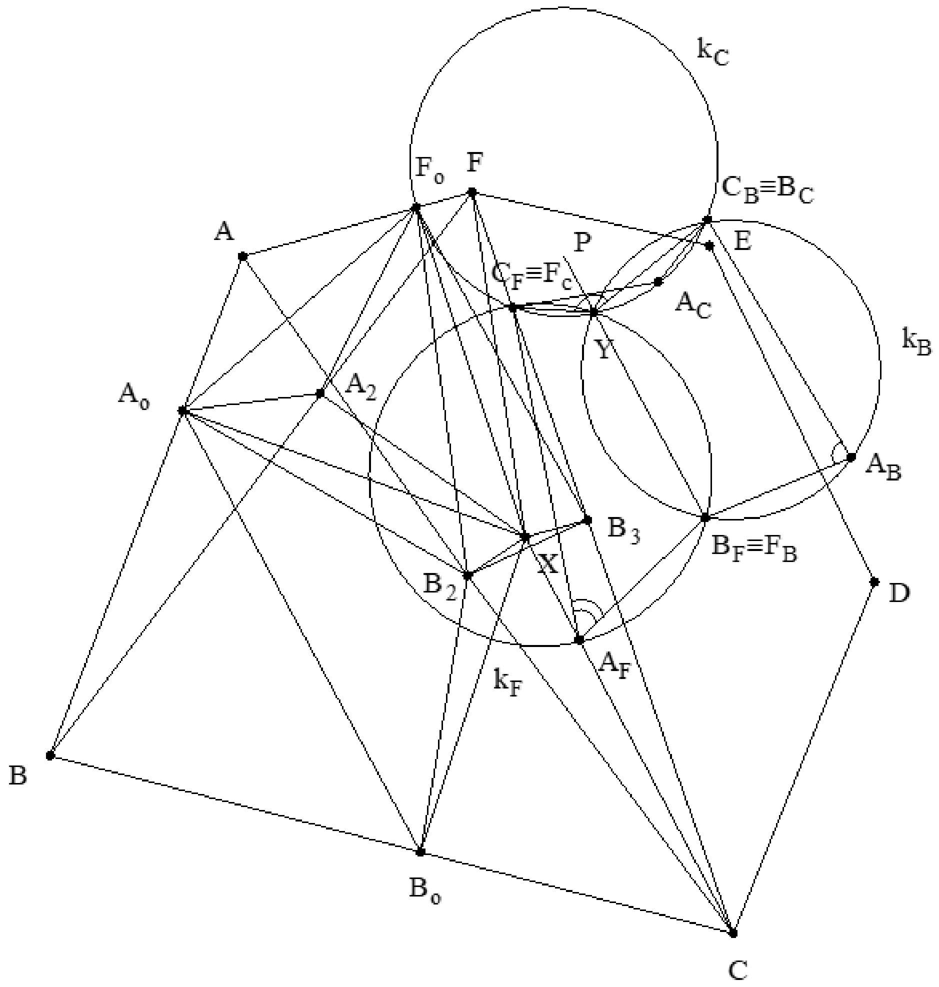


**FIGURE 10.** Shows **Theorem 3.**

Now let $Y$ be the second common point of the Simpson's circles $k_B$ and $k_F$. Let prove that $Y$ lies on $k_C$. Let $P$ be an arbitrary point on the continuation of the segment $F_BY$. From the inscribed in $k_F$ quadrilateral $YB_FA_FC_F \Rightarrow \sphericalangle F_CYP = \sphericalangle C_FA_FB_F$, and from the inscribed $YF_BA_BC_B$ in $k_B \Rightarrow \sphericalangle B_CYP = \sphericalangle C_BA_BF_B$.

Hence, using also the equations (6) – (8), we obtain:

$$\begin{aligned} &\sphericalangle F_CYB_C = \sphericalangle F_CYP + \sphericalangle B_CYP = \sphericalangle C_FA_FB_F + \sphericalangle C_BA_BF_B = \\ &= (\sphericalangle BXC - \sphericalangle BAC) + (\sphericalangle FXC - \sphericalangle FAC) = (\sphericalangle BXC + \sphericalangle FXC) - (\sphericalangle BAC + \sphericalangle FAC) = \\ &= (360^o - \sphericalangle BXF) - \sphericalangle BAF = \sphericalangle F_CA_CB_C, \end{aligned}$$

i.e.:

$$\sphericalangle F_C Y B_C = \sphericalangle F_C A_C B_C .$$

Hence the points $F_C, Y, A_C$, $B_C$ are concyclic, i.e. *Y* lies on the circle defined by $F_C, A_C$, $B_C$, but that's the Simpson's circle $k_C$. Therefore the Simpson's circles $k_C$, $k_B$ and $k_F$ have a common point *Y*. Analogously, the circles $k_A$, $k_F$ and $k_C$ have a common point. Since it is a common point of $k_F$ and $k_C$, it coincides with *Y*. Therefore $Y \in k_A \cap k_F \cap k_C \cap k_B$. In the same way $k_A$, $k_F$ and $k_D$ have a common point. Since it is a common point of $k_A$ and $k_F$, it coincides with *Y*. Therefore $Y \in k_A \cap k_F \cap k_C \cap k_B \cap k_D$. Finally, we similarly obtain that the circles $k_C$, $k_B$ and $k_E$ have a common point. Since it is a common point of $k_C$ and $k_B$, it is *Y*. Therefore $Y \in k_A \cap k_F \cap k_C \cap k_B \cap k_D \cap k_E$, i.e. all six Simpson's circles have a common point.

**Definition 3.** Let *X* be a point on the plane of the inscribed hexagon *ABCDEF*, not on its circumcircle. The common point *Y* of Simpson's circles for *X* with respect to the peripheral pentagons of *ABCDEF* we call the pedal image of *X* with respect to *ABCDEF* (Fig. 3).


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