Gramians Computation for Hyperbolic Distributed Parameter Systems

Kamen Perev
Systems and Control Department
Technical University of Sofia
Sofia, Bulgaria
email:kperev@tu-sofia.bg

***Abstract*—The paper considers the problem of gramians computation for linear hyperbolic distributed parameter systems. A special case of such systems is the vibrating string, which is described by wave partial differential equation. The weak solution of this equation is derived by applying the approach of time-space separation and using the Fourier method. This solution is transformed into a standard state space formulation by means of certain infinite dimensional matrices. The proposed system description is compared to a description based on a strongly continuous semigroup generated by a bounded system operator. Similarly to the finite dimensional case, it is shown that the derived weak solution consists of two parts. The zero input part is due to the initial conditions and participates in obtaining the observability gramian of the system. The zero state part is a consequence of the input signal effect and is used to compute the controllability gramian. The advantages of the presented approach for computing the gramians with respect to the standard semigroup approach is discussed.**

Keywords — distributed parameter system, hyperbolic partial differential equation, time-space separation, method of Fourier, vibrating string, controllability and observability gramians

I. INTRODUCTION

In many physical processes liable to control, the actual variables are functions of both time and position. The time dependence of these variables determines the dynamical nature of the physical process, and the position dependence determines the distributed character of the variables describing the physical phenomenon. From system theoretic point of view, such physical processes define the distributed parameter systems. Such systems are described by partial differential equations and their solution belongs to an infinite dimensional Hilbert space. The term distributed parameter is based on the fact that the corresponding solution reflects the distribution in space of the physical quantities. A specific feature of such systems is that they may lead to the existence of not rational transfer functions [1]. While finite dimensional systems, containing lumped parameters, are modeled by ordinary differential equations and rational transfer functions, the distributed parameter systems are modeled by partial differential equations and irrational transfer functions. A specific feature of these systems is the existence of infinite dimensional state space. The infinite dimensional nature of the state space is due to the fact that a distributed parameter system evolves over a continuum, where the state vector can be considered as a member of an infinite dimensional function space. Typical examples of distributed parameter systems are thermal, pneumatic, pressure and hydraulic systems, as well as systems describing chemical and biological processes, flexible mechanical structures, material microstructures and many others [2]. The wide spread of practical applications of such systems determines the need of their detailed and thorough study.

Hyperbolic partial differential equations find their application in modeling wave phenomena in physics. The propagation of waves in physics is related to the electromagnetic field distribution, acoustic and elastic waves dispense in different environments, the cases of standing waves of vibrating strings or membranes, flexible beams, resonating cavities and many others. A model reduction procedure for rapid and reliable solution of hyperbolic partial differential equations is proposed in [3]. The authors utilize a time-space data compression procedure for data processing, then apply a Petrov-Galerkin projection for computation of mapped solutions and finally speed up the process by online computation. A major approach for modeling distributed parameter systems is by using time-space separation. In [2], the authors give a presentation of the basic methods, which use time-space separation for developing the distributed parameter system models. The authors discuss partial differential equations models obtained from first principle knowledge, as well as the application of system identification methods for deriving distributed parameters descriptions. The main approach for solving the presented problems is the Fourier method for obtaining time and spatial function representations. A finite difference spatial discretization for preserving the geometrical structure of the underlying physical phenomena is proposed in [4]. The explored processes is 2-D wave and heat equations described in cylindrical coordinate system. The discretization structure is important part of the distributed parameter system modeling and control, since it presumes the application of appropriate numerical methods. A tutorial on frequency domain descriptions for distributed parameter systems is given in [1]. The authors present different transfer function models for the basic physical processes described by partial differential equations for different boundary conditions. Similar work, for different distributed parameter systems representations, both in time and frequency domain and for different initial and boundary conditions, is presented in [5]. Especially useful information regarding the input/output relation in time domain and directly related to the controllability system properties, are the Green functions for the underlying processes. The balanced truncation method for model order reduction of the semi discretized Stokes equation is presented in [6]. The relationship between input/output and internal stability and the concepts of stabilizability/detectability are extended from the finite dimensional to the infinite dimensional case in [7]. The generalization of the finite dimensional theory is made possible by assuming that the evolution of the state is governed by strongly continuous semigroup of bounded linear operators, where the state space is a Hilbert space, while the input and output spaces can be finite dimensional. The authors have proven that the stabilizability/detectability condition is

978-1-6654-2661-9/21/$31.00 ©2021 IEEE

necessary and sufficient for the existence of internally stabilizable output feedback compensation. Gramians computation for parabolic distributed parameter systems is presented in [8]. The solution of the partial differential equation is derived by applying the approach of time-space separation and by using the spectral method. The gramians are obtained as generalizations to the finite dimensional case, and are based on the solution decomposition of the linear parabolic partial differential equation.

The paper considers the problem of gramians computation for a linear hyperbolic distributed parameter system. The gramians are important energy characteristics of a given system, with the controllability gramian determining the system energy distribution at the input, and the observability gramian reflecting the energy at the system output. Gramians find application in system modeling through the method of balanced truncation model reduction. They can be used for controllability and observability analysis of a given system. Finally, gramians have important role in control system synthesis, by obtaining a control law of minimum energy.

We provide a new approach for computing the gramians, which is based on the solution of the hyperbolic partial differential equation. The main approach in the control literature for gramians derivation is also considered. This approach relies on infinite dimensional state space formulation of the partial differential equation problem and uses the properties of a C_0-semigroup generated by certain linear operator containing the derivative operator as its element. Our approach uses the weak solution of the hyperbolic partial differential equation, obtained by applying the principle of time-space separation. The spatial basis functions are determined by utilizing the Fourier method for the derived solution. By analogy to the finite dimensional case, the controllability and observability gramians of the distributed parameter system are obtained from the solution division into zero-input and zero-state parts. It is shown that the obtained gramians obey the semigroup properties of the partial differential equation operator. We examine the wave equation representing vibrating string, acoustic waves and flexible beam vibration and illustrate how the presented method for gramians computation can be applied for the vibrating string system.

II. MATHEMATICAL PRELIMINARIES ON HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS

We consider the general form of a second order partial differential equation with two independent variables:

$$a u_{tt} + b u_{tx} + c u_{xx} + d u_t + e u_x + g u = f, \qquad (1)$$

where $u = u(x,t)$ is the physical variable under consideration and $f = f(x,t)$ is the external force applied to the process. The partial differential equation reflects the distribution of a physical quantity and this is the reason to call the corresponding system a distributed parameter system. The equation (1) is homogeneous if the condition on the right hand side $f(x,t) \equiv 0$ is satisfied. If the discriminant $b^2 - 4ac > 0$, the equation (1) is of hyperbolic type. A main representative of the hyperbolic differential equation is the wave equation of a vibrating string. The one-dimensional nonhomogeneous wave equation can be written in the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \ 0 < x < L, t \geq 0, \qquad (2)$$

$u(x,t)$ is the deflection from the equilibrium state at time t in the position x along the string. The constant $\alpha = \frac{\tau}{\rho}$ is the string constant, where τ is the string tension and ρ is the linear mass density of the string. Equation (2) models different wave phenomena like acoustic plane waves, lateral vibrations in beams and others [9]. For the vibrating string, the boundary conditions are set to zero, i.e.:

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \qquad (3)$$

The initial conditions are set to the following functions:

$$u(x,0) = \varphi(x) \ \text{ and } \ \frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad (4)$$

Such kind of problem is called the problem of Cauchy. We assume that the external force applied to the string is given in the form $f(x,t) = b(x)v(t)$, where $b \in L_2(0,L)$ describes both the control and observation action, and after normalizations are introduced, the nonhomogeneous wave equation takes the form [1]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + b(x)v(t) \qquad (5)$$

with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Assuming velocity measurement at the output, we obtain:

$$y(t) = \int_0^L \frac{\partial u(x,t)}{\partial t} b(x)dx \qquad (6)$$

If instead of velocity measurement (6), we measure position, the output equation takes the form:

$$y(t) = \int_0^L u(x,t)b(x)dx \qquad (7)$$

The transfer function can be calculated by taking Laplace transforms on equations (5) and (6) and considering zero initial conditions, i.e. $\varphi(x) = \psi(x) = 0$. The resulting boundary value problem can be written in the form [1]:

$$\frac{d^2 U(x,s)}{dx^2} = s^2 U(x,s) - b(x)\Upsilon(s), \qquad (8)$$

where $U(x,s)$ and $\Upsilon(s)$ are the Laplace transforms of $u(x,t)$ and $v(t)$ correspondingly, with boundary conditions: $U(0,s) = 0$ and $U(L,s) = 0$ .Thus, equation (5) can be transformed into the form:

$$\frac{d}{dx}\begin{bmatrix} U \\ \frac{dU}{dx} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ s^2 & 0 \end{bmatrix}\begin{bmatrix} U \\ \frac{dU}{dx} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} b(x)\Upsilon(s) \qquad (9)$$

The matrix exponential of matrix $A = \begin{bmatrix} 0 & 1 \\ s^2 & 0 \end{bmatrix}$ with respect to the spatial variable x is given by the expression:

$$e^{Ax} = \begin{bmatrix} \cosh(sx) & \frac{1}{s}\sinh(sx) \\ s\sinh sx & \cosh(sx) \end{bmatrix} \qquad (10)$$

After selecting the function:

$$b(x) = \begin{cases} 1, & 0 \leq x \leq L/2 \\ 0, & L/2 < x \leq L \end{cases}$$

the transfer function of the vibrating string distributed parameter system is obtained as follows [1]:

$$G(s) = \frac{1}{2s} + \frac{2\cosh\left(\frac{s}{2}\right) - \cosh^2\left(\frac{s}{2}\right) - \cosh(s)}{s^2 \sinh(s)} \qquad (11)$$

The transfer function model is an input/output model in the frequency domain. We consider now the input/output model in time domain that is presented by the Green function. The Green function is a generalization of the impulse response for distributed parameter systems. We assume that, the hyperbolic differential equation under consideration is given as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \quad (12)$$

with initial conditions $u(x,0) = \varphi(x)$ and $\frac{\partial u}{\partial t}(x,0) = \psi(x)$. Assume the spatial interval as $0 \le x \le L$ and $t \ge 0$ .The Green function for (12) is given as follows [5]:

$$G(x,\xi;t) = \frac{2}{\pi\alpha}\sum_{k=1}^{\infty} \sin\frac{k\pi}{L}x \sin\frac{k\pi}{L}\xi \sin\frac{\alpha k\pi}{L}t \quad (13)$$

If the domain of the spatial variable is changed, then for $x \ge 0$ and $t \ge 0$, the Green function is obtained as [5]:

$$G(x,\xi;t) = \frac{1(x-\xi+\alpha t) - 1(x-\xi-\alpha t) - 1(x+\xi+\alpha t) + 1(x+\xi-\alpha t)}{2\alpha} \quad (14)$$

Finally, for an infinite length of the string, the Green function is obtained as [5]:

$$G(x,\xi;t) = \frac{1}{2\alpha} 1(\alpha t - |x-\xi|) \quad (15)$$

III. HYPERBOLIC GRAMIANS DERIVATION

The goal is to develop the concept of system gramians for distributed parameter systems with infinite dimensional state space, where the state evolves over a continuum and therefore, can be regarded as an element of infinite dimensional function space. The state equation for such systems can be written in the form [10]:

$$\frac{d}{dt}\begin{bmatrix} z \\ \frac{dz}{dt} \end{bmatrix} = A\begin{bmatrix} z \\ \frac{dz}{dt} \end{bmatrix} + Bv(t), \quad (16)$$

where $z(t) \in \mathcal{H}$, $\mathcal{H}$ is the infinite dimensional Hilbert space, A is not a matrix, but an operator acting on $\mathcal{H}$, i.e. $A: \mathcal{H} \to \mathcal{H}$ and $B: \Upsilon \to \mathcal{H}$, where Υ is the input space, which can be finite dimensional. For the vibrating string system, where the wave equation is given by the expression (5), the state equation can be written in the form (16), where the system operators are defined as follows:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2}(\cdot) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b(x) \end{bmatrix} \quad (17)$$

If we consider the distributed parameter system of a damped vibrating string, then the wave equation becomes a type of a telegraph equation, which is described as [1]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \epsilon \left\langle \frac{\partial u(\cdot,t)}{\partial t} \middle| b \right\rangle b(x) = \frac{\partial^2 u(x,t)}{\partial x^2} + b(x)v(t) \quad (18)$$

where ϵ is a small constant and under the same boundary conditions as in (5), then the system operators take the form:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2}(\cdot) & -\epsilon\langle(\cdot)|b\rangle b(x) \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ b(x) \end{bmatrix} \quad (19)$$

and since $z(t) = \frac{\partial u(\cdot,t)}{\partial t}$, the equation becomes nonlinear. The output equation, assuming velocity measurements at the output, takes the form:

$$y(t) = C\begin{bmatrix} z \\ \frac{dz}{dt} \end{bmatrix}, \quad (20)$$

where $C = [0 \quad \langle \cdot \,|b(\cdot)\rangle]$, where the integration in the inner product is performed in spatial domain with interval of integration $[0, L]$.

If we consider a simply supported flexible beam, the differential equation describing the beam deflection, is as follows [1]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^4 u(x,t)}{\partial x^4} = b(x)v(t), \ 0 < x < L, \ t \ge 0 \quad (21)$$

The boundary conditions for the flexible beam are determined as follows:

$$u(0,t) = 0, u(L,t) = 0, u_{xx}(0,t) = 0, u_{xx}(L,t) = 0 \quad (22)$$

The state equation for the flexible beam takes the form given in (16) with system operators, defined as follows:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{\partial^4}{\partial x^4}(\cdot) & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ b(x) \end{bmatrix} \quad (23)$$

The key concept for obtaining the solution of the infinite dimensional state equation, that generalizes the concept of state transition matrix, is the concept of a strongly continuous C_0 – semigroup.

Definition 1. (Strongly continuous semigroup) [10]:

A strongly continuous (C_{0-}) - semigroup $S(t)$ on the Hilbert space $\mathcal{H}$ is a family $S(t) \in \mathcal{L}(\mathcal{H},\mathcal{H})$ such that: *i)* $S(0) = I$; *ii)* $S(t)S(s) = S(t+s)$; *iii)* $\lim_{t\to 0} S(t)z = z, \forall z \in \mathcal{H}$.

Definition 2. (Infinitesimal generator) [10]:

The infinitesimal generator A of a C_{0-} - semigroup on $\mathcal{H}$ is defined by the expression:

$$Az = \lim_{t\to 0}\frac{1}{t}[S(t)z - z]$$

with domain $\mathcal{D}(A)$, that is the set of elements $z \in \mathcal{H}$ for which the limit exists.

The matrix exponential e^{At} is the special case of a semigroup, defined on a finite dimensional space. Similarly to the finite dimensional case, the following relation is satisfied in the infinite dimensional case:

$$\frac{d}{dt}S(t)z_0 = S(t)Az_0 \quad (24)$$

The homogeneous differential equation on the Hilbert space $\mathcal{H}$ can be written in the form:

$$\frac{dz(t)}{dt} = Az(t), \quad (25)$$

Therefore, the partial differential equation represented into a state space form, has the following solution:

$$z(t) = S(t)z_0 \quad (26)$$

In the nonhomogeneous case, the solution of equation (16) is obtained as follows:

$$z(t) = S(t)z_0 + \int_0^t S(t-\tau)Bv(\tau)d\tau \quad (27)$$

Then, the output signal is obtained as follows:

$$y(t) = CS(t)z_0 + C\int_0^t S(t-\tau)Bv(\tau)d\tau \quad (28)$$

By analogy to the finite dimensional case, we can define the controllability and observability maps, by using the state and output solutions from equations (27) and (28). The controllability map of the distributed parameter system over the interval $[0,T]$ is the bounded linear map $\mathcal{G}: \mathcal{H}([0,T],\Upsilon) \to \mathcal{H}$ defined by the relation [11]:

$$\mathcal{G}v = \int_0^T S(T-s)\, Bv(s)ds \tag{29}$$

The system is exactly controllable on $[0,T]$ if all points in $\mathcal{H}$ can be reached from the origin at time T, i.e. $range(\mathcal{G}) = \mathcal{H}$. The system is approximately controllable on $[0,T]$ if given a small ε, the origin can be steered to any state within a ε-neighborhood around it. In this case the reached states form a dense subspace within $\mathcal{H}$, or in other words, $\overline{range(\mathcal{G})} = \mathcal{H}$ [11]. Further, the controllability gramian of the distributed parameter system is defined as:

$$W_c(0,T) = \mathcal{G} * \mathcal{G}^* \tag{30}$$

where $\mathcal{G}^*$ is the adjoint of $\mathcal{G}$ and is defined as:

$$(\mathcal{G}^* z)(\tau) = B^* S^*(T-\tau)z \tag{31}$$

Finally, W_c is an element of $\mathcal{L}(\mathcal{H})$ and is defined as [11]:

$$W_c(0,T)z = \int_0^T S(\tau)BB^*S^*(\tau)zd\tau,\;\; z \in \mathcal{H} \tag{32}$$

In [11] is shown that, for the vibrating string distributed parameter system, the condition for exact controllability on the interval $[0,T]$ is satisfied, if and only if, there exists a constant $\gamma > 0$, such that $\int_0^T \|B^*S^*(\tau)z\|^2 d\tau \geq \gamma\|z\|^2$.

The observability map of the distributed parameter system on the interval $[0,T]$ is the bounded linear map $\mathcal{O}: \mathcal{H} \to \mathcal{H}([0,T];\mathcal{Y})$ defined by:

$$\mathcal{O}z = CS(\cdot)z \tag{33}$$

The system is exactly observable on $[0,T]$ if the initial state can be uniquely and continuously constructed from the knowledge of the output in $\mathcal{H}([0,T];\mathcal{Y})$, i.e. $\mathcal{O}$ is injective and its inverse is bounded on the range of $\mathcal{O}$ [11]. The system is approximately observable on $[0,T]$, if knowledge of the output in $\mathcal{H}([0,T];\mathcal{Y})$ determines the initial state uniquely, i.e. $ker(\mathcal{O}) = \{0\}$. The observability gramian on the interval $[0,T]$ is defined as:

$$W_o(0,T) = \mathcal{O}^*\mathcal{O} \tag{34}$$

Conditions for exact observability and approximate observability are discussed in [11]. The observability gramian W_o is an element of $\mathcal{L}(\mathcal{H})$ and can be written in the form:

$$W_o(0,T)z = \int_0^T S^*(\tau)C^*CS(\tau)zd\tau \tag{35}$$

Similarly to the controllability concept, it is shown in [11] that, for the vibrating string distributed parameter system, the condition for exact observability on the interval $[0,T]$ is satisfied, if and only if, there exists a constant $\gamma > 0$, such that $\int_0^T \|CS(\tau)z\|^2 d\tau \geq \gamma\|z\|^2$ for any $z \in \mathcal{H}$.

IV. GRAMIANS COMPUTATION FOR THE HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS

Equations (30) and (34) define the controllability and observability gramians for distributed parameter systems in operator form. Both gramians are obtained by using the concept of strongly continuous semigroup, generated by bounded operators containing spatial differentiation in their structure. This approach however, creates some difficulties concerned with computing the gramians. This is the reason to use the unifying framework of time-space separation of the partial differential equation solution. The principle of time-space separation is based on the possibility of Fourier series approximation of every continuous function describing the physical reality. Based on this principle, every time-space function can be expanded as the sum of products between time and space functions:

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)u_n(t) \tag{36}$$

We consider the vibrating string wave equation with presence of an external force:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),\;\; 0 < x < L,\; t \geq 0 \tag{37}$$

The boundary conditions are given as $u(0,t) = 0$ and $u(L,t) = 0$. The initial conditions are $u(x,0) = \varphi(x)$ and $\frac{\partial u}{\partial t}(x,0) = \psi(x)$. The goal is to obtain the weak solution for the hyperbolic partial differential equation. Let us consider the function of time $\eta(t)$ from class K [12], where:

$$K = C([0,\infty);\mathcal{H}) \cap C^{(1)}([0,\infty);\mathcal{H}) \tag{38}$$

Then, accepting the operator form $\Delta u = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}$, we can write:

$$\left\langle \frac{d^2u(t)}{dt^2} \middle| \eta(t) \right\rangle - \langle \Delta u(t) | \eta(t) \rangle = \langle f(t) | \eta(t) \rangle \tag{39}$$

We select a time moment T, such that $\eta(T) = 0$ and say that the function $u(t)$ is a weak solution of the vibrating string wave equation, if $u \in K$ and $u(0) = \varphi$, where $\eta(t)$ is an arbitrary function from the class K.

Therefore, the weak solution $u(x,t) = u(t)$ of the mixed problem for the vibrating string wave equation belongs to the class K and satisfies the equality [12]:

$$-\int_0^T \left\langle \frac{du(t)}{dt} \middle| \frac{d\eta(t)}{dt} \right\rangle dt + \int_0^T \langle \Delta u(t) | \eta(t) \rangle dt - \langle \psi | \eta(0) \rangle = \int_0^T \langle f(t) | \eta(t) \rangle dt \tag{40}$$

under the initial condition $u(0) = \varphi$. Further, we assume that $f(t) = f(x,t)$ is a function of class $C([0,\infty);\mathcal{H})$ and that, the solution of the equation (37) exists. For every $t \geq 0$, the weak solution is a member of the Hilbert space $\mathcal{H}(0,L)$ and therefore, can be expanded into a series with respect to every complete set of orthonormal basis functions in the same Hilbert space. This system of basis functions can be selected to be the system eigenfunctions ϕ_n, $n = 1,2,\cdots$ of the Laplace operator $\Delta u = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}$. We rewrite equation (36) in the form $u(t) = \sum_{n=1}^{\infty} u_n(t)\phi_n$, where $u_n(t) = \langle u(t) | \phi_n \rangle$. Thus, the series can be written in the form:

$$u(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, u_n(t) \frac{\phi_n}{\sqrt{\lambda_n}}, \tag{41}$$

where λ_n, $n = 1,2,\cdots$ are the eigenvalues of the Laplace operator, i.e. $\langle \Delta u(t) | \phi_n \rangle = \lambda_n u_n(t)$. The functions $\frac{\phi_n}{\sqrt{\lambda_n}}$ determine a complete orthonormal set of basis functions on $\mathcal{H}(0,L)$ and (40) gives a series representation of the weak solution $u(t)$. If in (40) we use $\eta(t) = (T-t)\phi_n$, we obtain the equation [12]:

$$\int_0^T u_n'(t)dt - T\langle \psi | \phi_n \rangle + \lambda_n \int_0^T (T-t)u_n(t)dt = \int_0^T (T-t)f_n(t)dt \tag{42}$$

where $f_n(t) = \langle f(t) | \phi_n \rangle$. After differentiation (42) with respect to T and making a substitution of the time variables, we obtain the equation:

$$u_n'(t) - \langle\psi|\phi_n\rangle + \lambda_n \int_0^t u_n(\tau)d\tau = \int_0^t f_n(\tau)d\tau \quad (43)$$

After differentiating with respect to t, we obtain the following differential equation [12]:

$$\frac{d^2u_n(t)}{dt^2} + \lambda_n u_n(t) = f_n(t) \quad (44)$$

with initial conditions $u_n(0) = \langle\varphi|\phi_n\rangle$ and $u_n'(0) = \langle\psi|\phi_n\rangle$.

The solution of equation (44) with the corresponding initial conditions is given as follows [12]:

$$u_n(t) = \langle\varphi|\phi_n\rangle \cos\sqrt{\lambda_n}t + \frac{\langle\psi|\phi_n\rangle}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\int_0^t \sin\sqrt{\lambda_n}(t-\tau)f_n(\tau)d\tau \quad (45)$$

Therefore, the weak solution of the vibrating string wave equation takes the form:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\langle\varphi|\phi_n\rangle \cos\sqrt{\lambda_n}t + \frac{\langle\psi|\phi_n\rangle}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\int_0^t \sin\sqrt{\lambda_n}(t-\tau)f_n(\tau)d\tau\right]\phi_n(x) \quad (46)$$

Since the equation is linear, its weak solution (46) has two parts:

i) the part that is due to the initial conditions and is the weak solution of the homogeneous equation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \alpha^2\frac{\partial^2 v(x,t)}{\partial x^2} = 0, \quad v|_{t=0} = \varphi, \ \left.\frac{dv}{dt}\right|_{t=0} = \psi$$

ii) the part that is due to the external input and is the weak solution of the nonhomogeneous equation under zero initial conditions:

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \alpha^2\frac{\partial^2 w(x,t)}{\partial x^2} = f(x,t), \ w|_{t=0} = 0, \left.\frac{dw}{dt}\right|_{t=0} = 0$$

The solutions of these two equations are obtained as follows:

$$v(x,t) = \sum_{n=1}^{\infty}\left[\langle\varphi|\phi_n\rangle \cos\sqrt{\lambda_n}t + \frac{\langle\psi|\phi_n\rangle}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t\right]\phi_n(x) \quad (47.1)$$

$$w(x,t) = \sum_{n=1}^{\infty}\frac{\phi_n(x)}{\sqrt{\lambda_n}}\int_0^t \sin\sqrt{\lambda_n}(t-\tau)f_n(\tau)d\tau \quad (47.2)$$

Therefore, the solution $u(x,t) = v(x,t) + w(x,t)$ consists of two parts, weak solutions of the above two differential equations (47.1) and (47.2).

The eigenvalues of the Laplace operator $\Delta u = \alpha^2\frac{\partial^2 u(x,t)}{\partial x^2}$ are obtained as [8]:

$$\lambda_n = \left(\frac{n\pi\alpha}{L}\right)^2, n = 1,2,\cdots \quad (48)$$

The corresponding eigenfunctions are $\phi_n = \sin\frac{n\pi}{L}x$, $n = 1,2,\cdots$, thus forming the orthonormal set of basis functions $\frac{\phi_n}{\sqrt{\lambda_n}} = \frac{L}{n\pi\alpha}\sin\frac{n\pi}{L}x$.

The initial conditions are computed as follows:

$$\langle\varphi|\phi_n\rangle = \frac{2}{L}\int_0^L \varphi(\xi)\sin\frac{n\pi\xi}{L}d\xi.$$

$$\langle\psi|\phi_n\rangle = \frac{2}{L}\int_0^L \psi(\xi)\sin\frac{n\pi\xi}{L}d\xi.$$

The existence of two initial conditions, as well as the form of equation (16) determines the order of this system to be two. Therefore, the C_0-semigroup of the vibrating string system is also of order two. We create a two by two block matrix $\Lambda_n(t) = \begin{bmatrix}\cos\frac{n\pi\alpha}{L}t & \sin\frac{n\pi\alpha}{L}t \\ -\sin\frac{n\pi\alpha}{L}t & \cos\frac{n\pi\alpha}{L}t\end{bmatrix}$, a two by one vector $Z_n = \begin{bmatrix}\langle\varphi|\phi_n\rangle \\ \frac{\langle\psi|\phi_n\rangle}{\sqrt{\lambda_n}}\end{bmatrix}$ and a two by one vector $\Phi_n = \begin{bmatrix}\sin\frac{n\pi}{L}x & 0\end{bmatrix}^T$, where $\sqrt{\lambda_n} = \frac{n\pi\alpha}{L}$.

The expression inside the sum in (47.1) can be written as $\Phi_n^T \cdot \Lambda_n(t) \cdot Z_n$. Further, we make the following construction by changing the index $n = 1,2,\cdots$ as follows: $\Lambda(t) = diag\{\Lambda_n(t)\}_{n=1}^{\infty}$, $Z = [Z_1^T \ Z_2^T \ \cdots \ Z_n^T \ \cdots]^T$, $\Phi = [\Phi_1^T \ \cdots \ \Phi_n^T \ \cdots]^T$. Thus, $v(x,t)$ from (47.1) can be obtained as:

$$v(x,t) = \Phi^T \cdot \Lambda(t) \cdot Z \quad (49)$$

We consider now equation (47.2). The input signal $f_n(t)$ is calculated as follows:

$$f_n(t) = \langle f(t)|\phi_n\rangle = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{n\pi\xi}{L}d\xi.$$

Next, we consider, that the input function is time-space separated as in (5) and can be expressed as $f(x,t) = b(x)\upsilon(t)$. So, we can write:

$$b_n = \langle b(x)|\phi_n\rangle = \frac{2}{L}\int_0^L b(\xi)\sin\frac{n\pi\xi}{L}d\xi \quad (50)$$

Similarly to the zero-input weak solution, we create a two by two block matrix $\Lambda_n(t) = \begin{bmatrix}\cos\frac{n\pi\alpha}{L}t & \sin\frac{n\pi\alpha}{L}t \\ -\sin\frac{n\pi\alpha}{L}t & \cos\frac{n\pi\alpha}{L}t\end{bmatrix}$, a two by one vector $B_n = \begin{bmatrix}0 \\ \frac{b_n}{\sqrt{\lambda_n}}\end{bmatrix}$, $\sqrt{\lambda_n} = \frac{n\pi\alpha}{L}$ and the two by one vector Φ_n is defined as above, i.e. $\Phi_n = \begin{bmatrix}\sin\frac{n\pi}{L}x & 0\end{bmatrix}^T$. Then, the expression inside the sum in (47.2) can be expressed as follows:

$$\Xi_n = \Phi_n^T \cdot \int_0^t \Lambda_n(t-\tau) \cdot B_n\upsilon(\tau)d\tau \quad (51)$$

Finally, $w(x,t)$ from (47.2) can be obtained as follows:

$$w(x,t) = \Phi^T \cdot \int_0^t \Lambda(t-\tau) \cdot B \cdot \upsilon(\tau)d\tau, \quad (52)$$

where $B = [B_1^T \ \cdots \ B_n^T \ \cdots]^T$, $\Lambda = diag\{\Lambda_n\}_{n=1}^{\infty}$ and Φ is defined as above.

For computation of the vibrating string distributed parameter system gramians, we apply the same approach as in [8]. This approach is based on the wave equation weak solutions (47.1) and (47.2). It implements in practice the operator equations (30) and (34) and gives direct realization of the expressions (32) and (35). In this sense, using expression (52) for the natural realization of expression (30), we obtain the controllability gramian for the vibrating string distributed parameter system on the interval $[0,T]$ as:

$$W_c(0,T) = \Phi^T \cdot \int_0^T \Lambda(t)BB^T\Lambda^T(t)dt \cdot \Phi. \quad (52)$$

Similarly, the observability gramian is computed by practical implementation of the operator equation (34). The observability gramian for the vibrating string distributed parameter system, computed on the interval $[0,T]$ takes the following form:

$$W_o(0,T) = \Phi^T \cdot \int_0^T \Lambda^T(t)\Lambda(t)dt \cdot \Phi. \qquad (53)$$

Remark 1. The so obtained gramians (52) and (53) are computed only for one spatial coordinate, namely x. In the case when the spatial coordinates are not fixed, but instead are changing on a whole interval $[0, L]$, we partition the interval into N spatial points, with $x_k = \left(k - \frac{1}{2}\right)\delta$, $\delta = \frac{1}{N}$, $k = 1,2,\cdots,N$. Then, we build the matrix $\mathrm{X} = [\phi(x_1) \quad \phi(x_2) \quad \cdots \quad \phi(x_N)]$ each column of which is $\phi(x_k) = [\phi_1(x_k) \quad \cdots \quad \phi_n(x_k) \quad \cdots]^T$ for spatial variable index $k = 1,2,\cdots,N$. Then, the gramians on the whole spatial frame, can be computed as:

$$W_c(0,T) = \mathrm{X}^T \cdot \int_0^T \Lambda(t)BB^T\Lambda^T(t)dt \cdot \mathrm{X}$$

$$W_o(0,T) = \mathrm{X}^T \cdot \int_0^T \Lambda^T(t)\Lambda(t)dt \cdot \mathrm{X}$$

Remark 2. The gramians are derived by using the infinite dimensional block-diagonal matrix $\Lambda(t)$ containing sine and cosine functions and consisting of the $[2 \times 2]$ block matrices $\Lambda_n(t) = \begin{bmatrix} \cos\frac{n\pi\alpha}{L}t & \sin\frac{n\pi\alpha}{L}t \\ -\sin\frac{n\pi\alpha}{L}t & \cos\frac{n\pi\alpha}{L}t \end{bmatrix}$, which are similarly equivalent to the matrices:

$$\widetilde{\Lambda}_n(t) = \begin{bmatrix} \cos\frac{n\pi\alpha}{L}t + j\sin\frac{n\pi\alpha}{L}t & 0 \\ 0 & \cos\frac{n\pi\alpha}{L}t - j\sin\frac{n\pi\alpha}{L}t \end{bmatrix} = \begin{bmatrix} e^{j\frac{n\pi\alpha}{L}t} & \\ & e^{-j\frac{n\pi\alpha}{L}t} \end{bmatrix}$$

where the above expression follows from the Euler's formula. Thus, it is clear that, matrix $\Lambda(t)$ satisfies the properties of a strongly continuous C_0 – semigroup. For practical computational purposes however, the infinite dimensional matrix $\Lambda(t)$ has to be truncated. The truncation introduces some errors of approximation, which have to be analyzed carefully.

Remark 3. The expressions for the gramians consist of two parts: one depending on spatial coordinates and the other one obtained from integration of certain infinite dimensional matrix on a finite interval of time. The second parts of the gramians are similar to the gramians for lumped parameter systems and similarly are positive definite or semidefinite matrices. If we fix the spatial coordinate, the obtained expressions are quadratic forms and show the immediate relation to energy interpretation of these system constructions in the finite and infinite dimensional cases.

V. CONCLUSION

This paper considers the problem of controllability and observability gramians computation for linear, time-invariant hyperbolic distributed parameters systems. The system under consideration is the vibrating string system. The weak solution of this system is obtained by applying the time-space separation approach, the method of Fourier and by using the spectral decomposition of the Laplace operator. Similarly to the case of lumped parameter systems, the weak solution of the wave equation is divided into two parts. The first part is due to the initial conditions of the hyperbolic partial differential equation. Based on this zero input solution, the observability gramian is derived. The second part is due to the external input effect. Based on this solution, the controllability gramian is obtained. The differences between the state space realizations in the distributed parameter and the lumped parameter cases are also discussed. It is shown that, the main difference evolves from the operator interpretation of the system dynamics. Based on this operator interpretation, the results for distributed parameter systems are derived as generalizations for the lumped parameter system results. The computational effectiveness of the obtained results is due to the successful application of the time-space separation principle, which is one of the basic principles for construction of system characteristics in the distributed parameter case. The obtained gramians are easy to compute and require elementary operations to derive. The approximations of the so obtained gramians can be used for solving many control theoretic problems, like model reduction, minimum energy control and many others.

ACKNOWLEDGMENT

This work was supported in part by the Research and Development Sector at the Technical University of Sofia.

REFERENCES

[1] R. Curtain and K. Morris, "Transfer functions of distributed parameter systems: A tutorial", Automatica, vol. 45, pp. 1101-1116, 2009.

[2] H.-X. Li and C. Qi, "Modeling of distributed parameter systems for applications – A synthesized review from time-space separation", Journal of Process Control, vol. 20, pp. 891-901, 2010.

[3] T. Taddei and L. Zhang, "Space-time registration-based model reduction of parameterized one-dimensional hyperbolic PDE's", Mathematical Modelling and Numerical Analysis, vol. 55, pp. 99-130, 2021.

[4] V. Trenchant, W. Hu, H. Ramirez and Y. L. Gorrec, "Structure preserving finite differences in polar coordinates for heat and wave equations", 9th Vienna International Conference on Mathematical Modelling, Vienna, pp. 624-629, 2018.

[5] A. G. Butkovsky, Green's functions and transfer functions handbook, Ellis Horwood, Chichester, New York, 1982.

[6] T. Stykel, "Balanced truncation model reduction for semidiscretized Stokes equation", Linear Algebra and its Applications, vol. 415, pp. 262-289, 2006.

[7] C. A. Jacobson and C. N. Nett, "Linear state-space systems in infinite-dimensional space: The role and characterization of joint stabilizability/detectability", IEEE Transactions on Automatic Control, vol.33, № 6, pp. 541-549, 1988.

[8] K. Perev, "Gramians computation for parabolic distributed parameter systems", International Conference on Automatics and Informatics, (ICAI'20), Varna, pp. 1 – 6, 2020.

[9] S. J. Farlow, Partial differential equations for scientists and engineers, J. Wiley & Sons, New York, 1982.

[10] K. Morris, "Control of systems governed by partial differential equations", in W. S. Levine edt., Control System Advanced Methods, pp. 67-1 – 67-37, 2011.

[11] R. Curtain and H. Zwart, Introduction to infinite-dimensional systems theory. A state-space approach, Springer, New York, 2020.

[12] S. G. Mikhlin, "Linear equations in partial derivatives", Moscow, Nauka, 1977.