Rational Function Approximation of the Relay with Hysteresis Nonlinear Element

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Abstract: This paper considers the problem of rational function approximation of the relay with hysteresis characteristic. The relay with hysteresis element has differential-based characteristic with memory, which is rate dependent and two-valued, involving the input signal velocity in its description. The presented model includes two ideal relay characteristics, which selection is depending on the input behavior. The rational function description is implemented for analytical approximation of the relay switching behavior. Both relay discontinuous jumps are approximated in terms of a hyperbolic tangent function, which on its own turn is replaced by continued fraction and Pade series expressions. The nearness of approximation is achieved by using one parameter descriptions for the hyperbolic tangent function. The errors of approximation of the ideal relay characteristic are also evaluated.

Keywords: relay with hysteresis characteristic, hyperbolic tangent function, continued fraction approximation, Pade series approximation

1. INTRODUCTION

Dynamical system modeling is related to the creation of simple, adequate and accurate mathematical descriptions for a given physical process for the purpose of its simulation and control. A special feature in modeling nonlinear control systems is the existence of hysteresis loops. Hysteresis is a phenomenon that is observed in many physical processes starting from mechanical devices and proceeding further to ferromagnetic materials and electrical circuits. The origin of hysteresis is in the existence of multiple state equilibria associated with system dynamics. The hysteresis nonlinearity can lead to performance degradation mainly in positioning accuracy of system performance, see Tan and Ram (2009). The difficulty in modeling hysteresis loops results from the existence of multivalued behavior that usually leads to energy storage in the system. At the same time, the energy storage is a frequent case of instability and self-sustained oscillations. The multivalued feature of hysteresis is due to the intrinsic memory effect, which is scale invariant. The memory effect is characterized by the property, that the response to particular changes is a function of preceding responses. Thus, corresponding to each input value, two output values of the hysteresis characteristic are possible and which one of these two values will occur depends on the history of the input. The memory property characterizes the rate-independent effect of the hysteresis loop, see for example Bermudez et al. (2020). This effect means that the branches of the hysteresis loop are determined only by the past extremum values of the input, while the speed of the input variations has no influence on branching. From the other side, the rate-dependent effect of the hysteresis loop indicates that the branching depends on the time scale variations. Thus, the output present value depends not only on the input present value but also on its velocity. The physical phenomenon of hysteresis has been observed in many fields of physics and engineering, just to mention friction, ferromagnetism, superconductivity, adsorption, mechanics, electrodynamics and others. The study of hysteresis physics and its modeling has a long history. The models used for hysteresis description are divided by Hassani et al. (2014) into two main groups: *i*) operator-based or static models, which use operators to describe the physical phenomenon and *ii*) differential-based or dynamic models, which use differential equations to model the hysteresis characteristic, see Macki et al. (1993). There exist four well known operator-based models to describe hysteresis loops: the Preisach model, the Krasnoselskii/Pokrovskii model, the Prandtl/Ishlinskii model and the Maxwell/Slip model, see for example Mayergoyz (1991). The Preisach model is one of the most popular operator-based models for describing hysteresis loops. The classical model is rate-independent and shows good performance characteristics for narrow frequency band and no load conditions of the explored materials. At the same time, the Preisach model has some disadvantages like difficulties in constructing the inverse characteristic, dependence of its accuracy from the collected amount of data and others. In order to overcome these difficulties, extensions from the classical Preisach model are obtained in the form of the Krasnoselskii/Pokrovskii model and the Prandtl/Ishlinskii model. The Krasnoselskii/Pokrovskii model developed by Krasnoselskii and Pokrovskii (1989), is an operator-based model, where the hysteresis loop description is derived as a linear combination of hysteresis operators. The model is successfully used to describe the hysteresis nonlinearity for shape memory alloys, shown in Zakerzadeh et al. (2011) and for continuous linearization in adaptive control, shown in Glen et al. (1998). Although the Krasnoselskii/Pokrovskii model improves the performance of the classical Preisach model, it still exhibits some difficulties in deriving the inverse hysteresis characteristic, which makes it difficult to use in real-time applications. The Prandtl/Ishlinskii model possesses a simpler mathematical structure than the former two models and is developed to overcome their disadvantages, finding application for hysteresis modeling in materials. This model has been successfully used in Krejci and Kuhnen (2001) for obtaining inverse models for a variety of hysteresis loops. The Prandtl/Ishlinskii model can be used to model asymmetric hysteresis loops as well, see Kuhnen (2003). The Maxwell/Slip model is an operator model that finds its application to express hysteresis nonlinearity in both mechanical and electrical systems. Its original application is in analyzing friction behavior for mechanical systems, see Hassani et al. (2014). The differential-based models use differential equations to describe the hysteresis phenomenon, see for example Visintin (1994). The Bouc/Wen model utilizes a nonlinear differential equation having the input derivative in its description. It is shown in Bellmunt et al. (2008) that, the Bouc/Wen model is especially suited for hysteresis loop approximation in control system applications. The main disadvantage of this model is that it is not invertible and needs some kind of compensation in the feedback structure. Another differential-based hysteresis model is the Duhem model. The main idea behind the Duhem model is to describe how the output variable changes its behavior when the input variable changes its direction. The Duhem model is rate-independent and uses polynomial approximation for the hysteresis loop estimation procedure, see Chen et al. (2011). Some other differential-based models are the Jiles/Atherton model, the Chua model, the Hodgon model and others, for more information see Nova and Zemanek (2010).

A special type of hysteresis characteristic representation is the relay with hysteresis nonlinear element, which is considered in Bagagiolo and Zoppello (2020) as representative of a delayed relay operator with thresholds. Relay with hysteresis elements play important role in nonlinear system analysis and design, and are associated with many practical control system problems. Control system elements having two-valued inertial characteristics are frequently part of numerous mechanical and electrical applications. It is shown by Bermudez et al. (2020) that in electrical engineering applications, the relay with hysteresis characteristic model can be used to estimate the energy losses in electrical machines for the purpose of electrical device design. Another very effective application of the relay with hysteresis characteristic is in automatic tuning of PID regulators. For the purpose of tuning, a relay with hysteresis is inserted in the closed loop and the PID regulator is temporarily disconnected. For many physical processes, the relay feedback initiates self-oscillations, which can be analyzed by the describing functions method. This kind of relay experiment can be easily automated and the regulator adjustment can be obtained very shortly, see Hägglund and Åström (2011). Relay with hysteresis characteristics find their application also in many algorithms and logic schemes, where switching with phase delay is taking part. For most of these cases, the analysis process is simplified if the relay with hysteresis characteristic is approximated by using analytic functions.

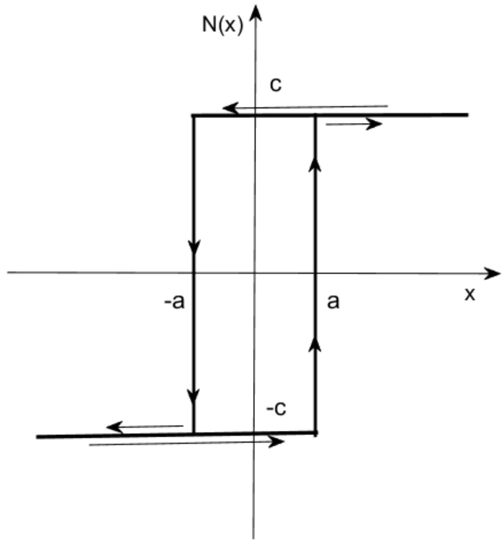

Fig. 1. The relay with hysteresis nonlinear characteristic

This paper considers the rational function approximation of the relay with hysteresis characteristic. A differential-based model involving the input signal velocity is developed assuming rate-dependent description. The switching behavior of the relay is approximated by using hyperbolic tangent functions. Finally, continued fraction and Pade series approximations of the exponential functions are used to estimate the hyperbolic tangent function thus, obtaining rational function approximation of the relay element. The closeness of the obtained characteristic curves in terms of the corresponding approximation errors is also discussed.

The main advantage of the presented approach is to use smooth function approximation for the jump behavior of the relay with hysteresis characteristic.The differentiability property of the approximation function allows using the presented model as part of different nonlinear methods, where the characteristic differentiability is a major requirement for their employment. For example, the Lyapunov based approach demands differentiability of Lyapunov functions, which condition prescribes the approximation model quite suitable for such applications.

2. ANALYTIC MODEL DEVELOPMENT FOR THE RELAY WITH HYSTERESIS CHARACTERISTIC

We consider the nonlinear relay with hysteresis characteristic shown in fig.1.

The usual mathematical expression describing this nonlinear element is given as follows:

$$N(x) = \begin{cases} c, & x \geq a, \ \dot{x} > 0 \\ -c, & x < a, \ \dot{x} > 0 \\ c, & x > -a, \ \dot{x} < 0 \\ -c, & x \leq -a, \ \dot{x} < 0 \end{cases} \tag{1}$$

It is clearly seen that this characteristic is two-valued in the interval $-a < x < a$. In this interval, the output variable depends on the derivative of the input, i.e. for $\dot{x} > 0$, $N(x) = -c$ and for $\dot{x} < 0$, $N(x) = c$. At the points

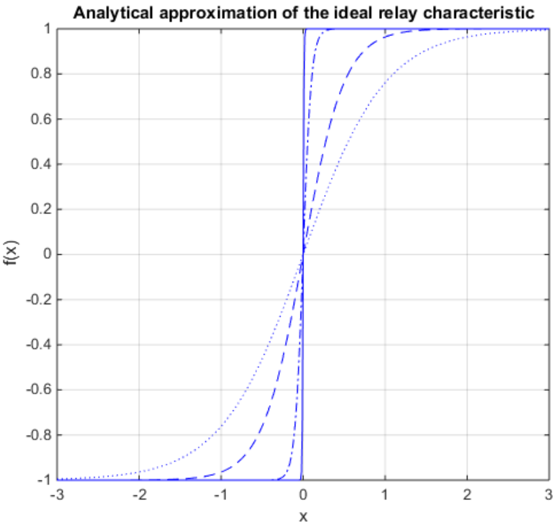

Fig. 2. Ideal relay approximation: $f(x) = \tanh(Ax)$ $A = 1.0$ ...., $A = 2.0$ - - -, $A = 10.0$ -.-.-, $A = 100.0$ —

$x = \pm a$ there is a jump in the output value and the relay element exhibits switching behavior. Expression (1) can be written in the form:

$$N(x) = \frac{c}{2}\{sign(x-a)[1+sign(\dot{x})] + sign(x+a)[1-sign(\dot{x})]\}. \tag{2}$$

From the above expression is clear that, if $\dot{x} > 0$, the second term in (2) is zero and the nonlinear relay with hysteresis element behaves like an ideal relay element switching at $x = a$. From the other side, if $\dot{x} < 0$, the first term in (2) is zero and the nonlinear relay with hysteresis element behaves like an ideal relay element switching at $x = -a$. Therefore, in order to develop an analytical model for the relay with hysteresis element, it is necessary to develop analytical expressions for the ideal relay model, i.e. for the $sign(x)$ function. The ideal relay is a non-memory nonlinear element characteristic, which can be represented by the function:

$$N(x) = c \cdot sign(x) = \begin{cases} c, & x > 0 \\ 0, & x = 0, \\ -c, & x < 0 \end{cases} \tag{3}$$

There is no convergent Taylor series for the function (3) at $x = 0$. Similarly, orthogonal polynomial approximations for such functions have very slow rate of convergence as well. One possible solution of the approximation problem for the $sign(x)$ function (3) is the usage of gate functions, presented in Schetzen (1989). We consider an analytical approximation for the $sign(x)$ function as a gate, expressed by the hyperbolic tangent function: $f(x) = \tanh(Ax)$, where A is a parameter. Figure 2 presents different plots of $f(x) = \tanh(Ax)$ for different values of the parameter A, like $A = 1, 2, 10, 100$.

For $A = 1$, the function $f(x) = \tanh(Ax)$ (the dotted line ....) clearly deviates from the ideal relay characteristic. Increasing the parameter A ($A = 2$, the dashed line - -) and ($A = 10$, the dash-dotted line -.-.), the approximation gets closer and closer to the ideal relay. Finally for $A = 100$ (the solid line —), the approximation of the ideal relay almost coincides with the true characteristic and the difference between these two characteristics is insignificant. The approximation error can be computed by using the following error expressions:

- The mean square approximation error:

$$\Delta_{mse} = \left\{ \frac{1}{b-a} \int_a^b [N(x) - f(x)]^2 \, dx \right\}^{\frac{1}{2}}$$

- The average relative approximation error:

$$\Delta_{are} = \frac{1}{b-a} \frac{1}{N_{max}} \int_a^b |N(x) - f(x)| \, dx,$$

where $N_{max} = \max_{a \le x \le b} N(x)$

- The maximal absolute approximation error:

$$\Delta_{mae} = \max_{a \le x \le b} |N(x) - f(x)|$$

For all three errors presented above, $[a, b]$ is the interval of observation, where the data is collected. The errors of approximation of the ideal relay characteristic by the function $f(x) = \tanh Ax$, using a step of $\delta = 0.01$ on the interval $[-3, 3]$, for all three error criteria and different values of the parameter A are presented in the table shown below.

Table 1. Approximation errors for different A's

	Δ_{mse}	Δ_{are}	Δ_{mae}
A = 1	0.3562	0.2282	0.99
A = 2	0.2502	0.1137	0.98
A = 10	0.1061	0.0214	0.90
A =100	$1.392 \cdot 10^{-2}$	$0.932 \cdot 10^{-3}$	0.2384

The obtained results from table 1 clearly show that by increasing the parameter value of A, the error of approximation decreases for all three criteria. As higher is the value of A, as smaller is the value of the approximation error. By increasing A, the mean square approximation error and the average relative approximation error criteria give similar results as concerned rate of convergence. Slightly higher values for the error of approximation gives the maximal absolute error criterion.

Our next problem is to approximate the hyperbolic tangent function in terms of rational functions of the argument. The computation of the hyperbolic tangent function, available in the computer libraries in FORTRAN or C languages are presented in Beebe (2017). It is shown that, there exist four different regions, where the $\tanh(x)$ function is evaluated. Since the $\tanh(x)$ is bounded for all real x, the limit value of the function for $x = \infty$ is equal to one. This case is suitably exploited in the computer algorithms for the function evaluation, and for values of the argument $x > x_{large} = 19.06155$, the function $\tanh(x)$ acquires the value of one. The selection of x_{large} is based on the assumption that for argument values $x > x_{large}$, the expression $\frac{2}{\exp(2x)+1}$ will be negligible relative to one. Thus, for $x > x_{large}$, $\exp(2x) \gg 1$ and therefore, $\frac{2}{\exp(2x)} < B^{-t-1}$, where B is the machine base with t fractional digits, see Beebe (2017). If we accept as machine base $B = 2$, the above inequality can be simplified as $\frac{(t+2)\ln(2)}{2} < x$. In the limit case, transforming the inequality to equality, and assuming $t = 53$ digits for double precision, we obtain $x_{large} = 19.06155$. In the derivations to follow, we divide the interval for computing $\tanh(x)$ in two parts. In the first interval for $|x| \leq x_{large}$, the hyperbolic tangent function is computed by the expression $\tanh(x) = 1 - \frac{2}{\exp(2x)+1}$. In the second interval for $|x| > x_{large}$, the value of the hyperbolic tangent function is considered as one. It is clear that, in order to compute the $\tanh(x)$, we need to compute the exponential function $\exp(2x)$. Therefore, the rational approximation of the $sign(x)$ function reduces to rational function approximation of $\exp(2x)$.

3. RATIONAL FUNCTION APPROXIMATION OF THE EXPONENTIAL FUNCTION

There exist different methods for rational function approximation of the exponential function. We consider here two of these methods: the continued fraction method and the Pade approximation method. The continued fraction method is based on using continued fractions for representation of the exponential function, which gives a convenient mechanism for its computation. First Euler has obtained the continued fraction expansion of $\exp(x)$ and showed that it is convergent for any real and complex value of the argument. The advantage of using continued fractions is the existence of recurrent procedure for computing the rational function parameters. The approximation error decreases by increasing the approximation index, i.e. the sum of the numerator and denominator polynomials order. All computed approximations having the same index, require the same number of elementary computations and have almost the same precision. The main disadvantage of continued fractions is the possible loss of accuracy when subtracting nearly equal quantities. The continued fraction expansion of the exponential function $\exp(x)$ can be presented as follows, see Demidovich and Maron (1987):

$$\exp(x) = \left[0;\ \frac{1}{1},\ \frac{-2x}{2+x},\ \frac{x^2}{6},\ \frac{x^2}{10},\ \frac{x^2}{14},\ \dots\ \frac{x^2}{4n+2},\ \dots\right] \tag{4}$$

The above continued fraction can be rewritten in terms of two variables ratios as:

$$\exp(x) = \left[a_0,\ \frac{b_1}{a_1},\ \frac{b_2}{a_2},\ \frac{b_3}{a_3},\ \frac{b_4}{a_4},\ \frac{b_5}{a_5},\ \frac{b_6}{a_6},\ \dots\right] \tag{5}$$

Using expressions (4) and (5), the exponential function can be evaluated as a ratio of two polynomials $P_n(x)$ and $Q_n(x)$, which satisfy the following recurrence relations, see Demidovich and Maron (1987):

$$P_{k+1}(x) = a_{k+1}P_k(x) + b_{k+1}P_{k-1}(x) \tag{6}$$

$$Q_{k+1}(x) = a_{k+1}Q_k(x) + b_{k+1}Q_{k-1}(x), \quad k = 1, 2, \dots \tag{7}$$

where the initial polynomials are defined by the expressions:

$$P_0(x) = Q_0(x) = 0, \quad P_1(x) = Q_1(x) = 1 \tag{8}$$

Using the recurrence relations (6), (7) and (8), we obtain the polynomials $P_9(x)$ and $Q_9(x)$ as:

$$P_9(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \dots \alpha_7 x^7 + x^8$$

$$Q_9(x) = \alpha_0 - \alpha_1 x + \alpha_2 x^2 - \alpha_3 x^3 + \dots - \alpha_7 x^7 + x^8$$

where the coefficients α_i, $i = 1, 2, \dots, 7$ acquire the following values:

$$\alpha_0 = 518918400, \quad \alpha_1 = 259459200, \quad \alpha_2 = 60540480,$$
$$\alpha_3 = 8648640, \quad \alpha_4 = 831600, \quad \alpha_5 = 211440,$$
$$\alpha_6 = 2520, \quad \alpha_7 = 272$$

and the exponential function is approximated by $\exp(x) \approx \frac{P_9(x)}{Q_9(x)}$. Using the continued fraction method, we obtain for $x = 1$, $e \approx 2.7182818284590 45$ which is evaluated with error $\varepsilon = O(10^{-16})$. Applying the continued fraction method for the exponential function evaluation, we obtain the following approximation for the hyperbolic tangent function for the values $|x| < x_{large} = 19.06155$ by the rational function expression:

$$\tanh(x) \approx 1 - \frac{2}{\frac{P_9(2x)}{Q_9(2x)} + 1}. \tag{9}$$

The other method, which we use for obtaining rational function approximation of the exponential function $\exp(x)$ is the Pade series method, see Baker and Graves-Moris (1981). The Pade approximation method is part of the shift operator methods like the Laguerre, Kautz and others, for rational function approximation of exponential functions. It is especially suited to work with in the complex domain, where Pade approximations are frequently used to approximate the time delay operator, see for example Perev (2011). The Pade approximation method has good convergent properties, but requires solving a linear system of algebraic equations for computing the rational function parameters. Pade series is based on the Taylor series presentation of the approximated function. The Taylor series of the exponential function is given by the expression:

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots \tag{10}$$

The Pade series approximation is a rational function, which is developed from the Taylor series and contains as many parameters as the corresponding parameters in the Taylor series. For example, for a Taylor series expansion with N parameters, a Pade series representation can be obtained as:

$$P_{L,M}(x) = \frac{a_0 + a_1 x + a_2 x^2 + \dots + a_L x^L}{1 + b_1 x + b_2 x^2 + \dots + b_M x^M} \tag{11}$$

where $L + M + 1 = N$. In (11), we have $L + 1$ parameters in the numerator and M parameters in the denominator, which can be computed from the coefficients of the Taylor series, satisfying the following relation, see Baker and Graves-Moris (1981):

$$\sum_{i=0}^{\infty} c_i x^i \approx \frac{a_0 + a_1 x + a_2 x^2 + \dots + a_L x^L}{1 + b_1 x + b_2 x^2 + \dots + b_M x^M} \tag{12}$$

The Pade series parameters computation is straightforward as:

$$(c_0 + c_1 x + c_2 x^2 + \dots)(1 + b_1 x + \dots + b_M x^M) \tag{13}$$
$$= (a_0 + a_1 x + \dots + a_L x^L)$$

For example, if the number of the available Taylor series coefficients is $N = 15$, the following Pade series $P_{7,7}(x)$ can be computed by satisfying the relations:

$$
\begin{array}{rcl}
c_0 \cdot 1 & = & a_0, \\
c_0 b_1 + c_1 & = & a_1, \\
c_0 b_2 + c_1 b_1 + c_2 & = & a_2, \\
\vdots & & \vdots \\
c_0 b_7 + c_1 b_6 + \ldots + c_6 b_1 + c_7 & = & a_7, \\
c_1 b_7 + c_2 b_6 + \ldots + c_7 b_1 & = & -c_8, \\
c_2 b_7 + c_3 b_6 + \ldots + c_8 b_1 & = & -c_9, \\
c_3 b_7 + c_4 b_6 + \ldots + c_9 b_1 & = & -c_{10}, \\
\vdots & & \vdots \\
c_7 b_7 + c_8 b_6 + \ldots + c_{13} b_1 & = & -c_{14}.
\end{array} \quad (14)
$$

The coefficients b_i, $i = 1, 2, \ldots, 7$ can be obtained by solving the system of linear algebraic equations as follows:

$$
\begin{bmatrix} c_1 & c_2 & c_3 & \ldots & c_7 \\ c_2 & c_3 & c_4 & \ldots & c_8 \\ c_3 & c_4 & c_5 & \ldots & c_9 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_7 & c_8 & c_9 & \ldots & c_{13} \end{bmatrix} \cdot \begin{bmatrix} b_7 \\ b_6 \\ b_5 \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} c_8 \\ c_9 \\ c_{10} \\ \vdots \\ c_{14} \end{bmatrix} \quad (15)
$$

The coefficients a_i, $i = 1, 2, \ldots, 7$ can be obtained by using the first seven equations of (14). Solving equations (14) and (15), we obtain the computed Pade series approximation model $P_{7,7}(x)$ for the exponential function $\exp(x)$, which is given by the following rational function:

$$
P_{7,7}(x) = \frac{1 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \ldots + \beta_6 x^6 + \beta_7 x^7}{1 - \beta_1 x + \beta_2 x^2 - \beta_3 x^3 + \ldots + \beta_6 x^6 - \beta_7 x^7}
$$

where

$$
\beta_1 = 0.5 \quad \beta_2 = 0.1154 \quad \beta_3 = 0.016 \quad \beta_4 = 0.0015
$$
$$
\beta_5 = 8.74 \cdot 10^{-5} \quad \beta_6 = 3.24 \cdot 10^{-5} \quad \beta_7 = 5.78 \cdot 10^{-8}
$$

Using the Pade series approximation method for $\exp(x) \approx P_{7,7}(x)$, we obtain the exponential function when $x = 1$, as $e \approx 2.718281828458230$ which is evaluated with error $\varepsilon = O(10^{-12})$.

4. RATIONAL FUNCTION APPROXIMATION OF THE HYPERBOLIC TANGENT FUNCTION

We consider now the rational function approximation of $\tanh(x)$. For the values of the input $|x| \leq x_{large} = 19.06155$, the hyperbolic tangent function takes on the form $\tanh(x) = 1 - \frac{2}{\exp(2x)+1}$ and for $|x| > x_{large}$, we consider the function value as $\tanh(x) = \pm 1$. We assume rational function representation of the exponential function and thus, approximating the hyperbolic tangent function as $\tanh(x) = 1 - \frac{2}{G(2x)+1}$, where for continued fraction approximation $G(2x) = \frac{P_9(2x)}{Q_9(2x)}$ and for Pade series approximation $G(2x) = P_{7,7}(2x)$. For the purpose of $sign(x)$ function approximation, we consider the hyperbolic tangent function with scaled argument $\tanh(Ax) \approx 1 - \frac{2}{\frac{P_9(2Ax)}{Q_9(2Ax)}+1}$ for the continued fraction approximation and $\tanh(Ax) \approx 1 - \frac{2}{P_{7,7}(2Ax)+1}$ for the Pade series approximation. Fig.3 shows the $\tanh(Ax)$ function and its approximation in terms of continued fractions and Pade series for the values of the argument $|x| \leq 0.1$ and for the values of the parameter $A = 30$ and $A = 100$.

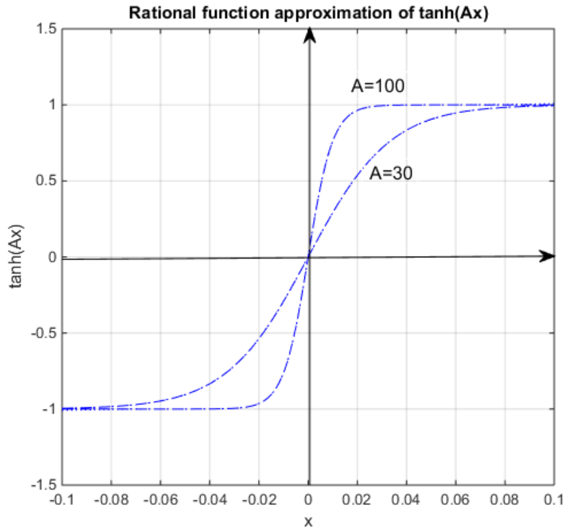

Fig. 3. Rational function approximation of $\tanh(Ax)$: continued fractions $(--)$, Pade series $(-.-)$ (both curves are indistinguishable)

From fig.3 is clearly seen that, all three curves: $\tanh(Ax)$ (...), the continued fraction approximation of $\tanh(Ax)$ (–) and the Pade series approximation of $\tanh(Ax)$ (-.-.), for both cases $A = 30$ and $A = 100$, coincide. This closeness of the results can be explained with the good approximation properties of the rational function representations $G(2Ax) = \frac{P_9(2Ax)}{Q_9(2Ax)}$ and $G(2Ax) = P_{7,7}(2Ax)$ for the exponential function $\exp(2Ax)$. A special attention should be placed on the x-axis scaling. This type of approximation is valid only on a limited time interval, where $2Ax \leq x_{large} = 19.06155$. Next we consider the problem of the rational function approximation for the relay with hysteresis characteristic. From expression (2) is evident the existence of four $sign(\cdot)$ functions for the relay with hysteresis model. Two of these functions depend on the argument derivative $sign(\dot{x})$, and two of them depend on the argument itself $sign(x)$. When the argument x increases, the relay with hysteresis characteristic is equivalent to the relay characteristic $N(x) = c \cdot sign(x - a)$, where a is the displacement from zero, and when the argument decreases, the relay with hysteresis characteristic is equivalent to the relay characteristic $N(x) = c \cdot sign(x + a)$. We choose $a = 1.0$, $A = 100.0$ and $c = 1.0$. Fig.4 shows the relay with hysteresis characteristic, obtained by rational function approximation in terms of continued fractions and Pade series.

As can be expected, both approximations, the continued fraction and the Pade series representations almost perfectly coincide with the relay with hysteresis characteristic. Both approximation curves exhibit switching at the points $x = \pm a$, demonstrating analytical description in terms of rational function representation for the jump phenomenon.

5. CONCLUSION

The paper considers the problem of rational function approximation of the relay with hysteresis nonlinear characteristic. The presented relay with hysteresis model is rate dependent, two-valued, nonlinear differential-based model

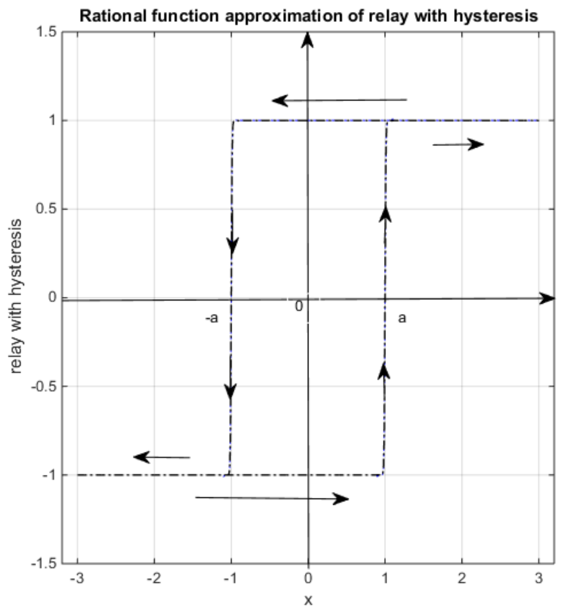

Fig. 4. Rational function approximation of relay with hysteresis by continued fractions and Pade series

involving the input signal velocity in its description. We developed an analytical model, where the relay jump behavior is approximated by the hyperbolic tangent function, which on its own turn is represented by continued fractions and Pade series estimation of the exponential function. The presented model contains one parameter, which value determines the proximity of the approximation to the true characteristic. It is shown that in close neighborhood around the relay discontinuous jump, the approximation functions perform well and give accurate results. The approximation errors of the rational function representations are also discussed. It is shown experimentally, that both methods for rational function approximation of the exponential function, continued fractions and Pade series, give almost the same results in terms of computing complexity and accuracy.

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