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Analysis of a 3-Conductor Transmission Line with Nonlinear Resistive Loads

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***Abstract*—This paper analyzes the electromagnetic compatibility characteristics of lossless transmission lines terminated by nonlinear loads suggested by C. Paul. The nonlinearities of the resistive loads are of polynomial type. We examine the mutual interaction between the two lines without neglecting the impact of the receptor line, which enables a more general mathematical model contrasted to the C. Paul model. We articulate a mixed problem for a system corresponding to 3-conductor transmission line. After that the hyperbolic system is transformed to a diagonal form and it is reduced to an initial value problem on the boundary. We obtained a system of two functional equations and two neutral equations for four unknown functions.**

Keywords— lossless transmission lines, nonlinear resistive loads, mixed problem

I. Introduction

Electromagnetic compatibility (EMC) aspects of VLSI systems and applications are studied extensively in the literature [1]-[10]. In this paper we elaborate an EMC model of a printed circuit board (PCB) based on 3-conductor transmission line using C. Paul results [11]. As distinct from the considerations in [11], we propose a broader treatment to find a solution to the system describing the 3-conductor transmission line problem; the transmission lines are terminated by nonlinear resistive loads as distinct to our previous paper [12] where we study linear loads.

In particular, we formulate a hyperbolic system modeling the behavior of 3-conductor transmission line terminated by nonlinear resistive loads (Fig.1). Then we transform it to a diagonal form using the method from [13]; we transform also the initial and boundary conditions. The reduced system consists of two functional equations and two equations of neutral type [14] on the boundary. The mixed problem for the diagonal system is reduced to an initial problem on the boundary.

The ground symbol in Fig. 1 is a denotation for the line voltages reference of the conductor. It is a PCB land in our case, but it may also be an endless ground plane, an overall shield, a wire, etc. The additional two conductors are PCB lands, but they might be of a diverse type. It is presumed that the line behaves as a uniform line (cf. [9], [10]). The line is supposed to be lossless in the sense that we have perfect conductors. The adjacent environment is also presumed to be lossless – it might be inhomogeneous similarly to the case of a PCB.

The upper circuit in Fig. 1 is referred to as generator circuit where the conductor with respect to the reference conductor is operated by a source having open-circuit voltage $U_S(t)$, source resistance R_S ; it is terminated in a resistive load R_L. The bottom circuit conductor with respect to the reference conductor circuit is referred to as "receptor". It is ended with a resistive load R_{NE} at the near end and with a resistive load R_{FE} at the far end. The current and voltage of the generator circuit create both magnetic and electric fields and they interrelate with the receptor circuit creating crosstalk voltages at the circuit terminals so the generator circuit perturbates the receptor circuit.

The reference conductor voltages $u_k(x,t)$ (k=1,2) and the currents of each circuit $i_k(x,t)$ (k=1,2) are dependent on position x and time t.

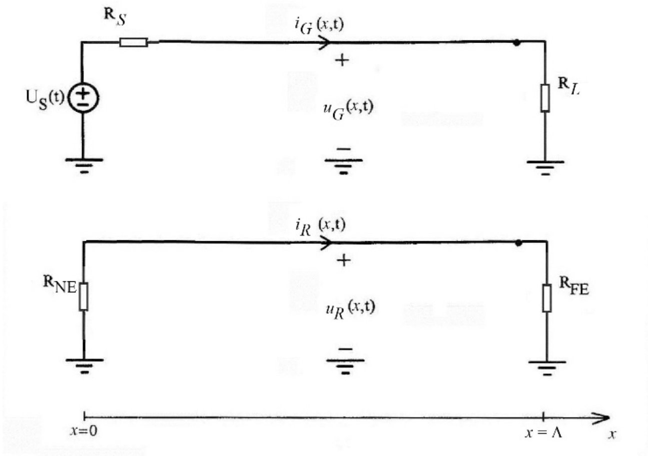

Fig. 1. Schematic of the three-conductor transmission line.

In agreement with the TEM mode of propagation (cf. [1]-[10]) we get the following mixed problem (initial-boundary value) for the hyperbolic system describing the behavior of the 3-conductor transmission line terminated by nonlinear resistive loads

$$\left|\begin{array}{l}\dfrac{\partial u_G(x,t)}{\partial x}+L_G\dfrac{\partial i_G(x,t)}{\partial t}=-L_m\dfrac{\partial i_R(x,t)}{\partial t}\\ \dfrac{\partial i_G(x,t)}{\partial x}+(C_G+C_m)\dfrac{\partial u_G(x,t)}{\partial t}=C_m\dfrac{\partial u_R(x,t)}{\partial t}\\ \dfrac{\partial u_R(x,t)}{\partial x}+L_R\dfrac{\partial i_R(x,t)}{\partial t}=-L_m\dfrac{\partial i_G(x,t)}{\partial t}\\ \dfrac{\partial i_R(x,t)}{\partial x}+(C_R+C_m)\dfrac{\partial u_R(x,t)}{\partial t}=C_m\dfrac{\partial u_G(x,t)}{\partial t}\end{array}\right. \tag{1}$$

with these boundary conditions

$$\left.\begin{array}{l}u_G(0,t)=U_S(t)-R_S i_G(0,t)\\ u_R(0,t)=-R_{NE} i_R(0,t)\end{array}\right\}\text{at the near end}$$

$$\left.\begin{array}{l}C_1\dfrac{du_G(\Lambda,t)}{dt}=i_G(\Lambda,t)-f_1(u_G(\Lambda,t))\\ C_2\dfrac{du_R(\Lambda,t)}{dt}=i_R(\Lambda,t)-f_2(u_R(\Lambda,t))\end{array}\right\}\text{at the far end} \tag{2}$$

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where $t \geq 0$ and the following initial conditions (at $t = 0$)

$$\begin{aligned} &u_G(x,0)=u_{G0}(x),\;\; u_R(x,0)=u_{R0}(x),\;\; x\in[0,\Lambda] \\ &i_G(x,0)=i_{G0}(x),\;\;\; i_R(x,0)=i_{R0}(x),\;\;\; x\in[0,\Lambda] \end{aligned} \tag{3}$$

In (2) f_1 and f_2 are polynomials. Functions $i_1 = f_1(u)$, $i_2 = f_2(u)$ are the *IV* characteristics of the nonlinear resistive elements; the nonlinearities are of polynomial type, that is

$$f_n(u) = \sum_{k=1}^{m} g_k^n u^k \;\; (n = 1,2).$$

The above system (1) is rewritten in the form

$$\begin{cases} (C_G+C_m)\dfrac{\partial u_G(x,t)}{\partial t} - C_m \dfrac{\partial u_R(x,t)}{\partial t} + \dfrac{\partial i_G(x,t)}{\partial x} = 0 \\ -C_m \dfrac{\partial u_G(x,t)}{\partial t} + (C_R+C_m)\dfrac{\partial u_R(x,t)}{\partial t} + \dfrac{\partial i_R(x,t)}{\partial x} = 0 \\ L_G \dfrac{\partial i_G(x,t)}{\partial t} + L_m \dfrac{\partial i_R(x,t)}{\partial t} + \dfrac{\partial u_G(x,t)}{\partial x} = 0 \\ L_m \dfrac{\partial i_G(x,t)}{\partial t} + L_R \dfrac{\partial i_R(x,t)}{\partial t} + \dfrac{\partial u_R(x,t)}{\partial x} = 0 \end{cases} \tag{4}$$

and the denotations are introduce

$$u_1(x,t)=u_G(x,t); \quad u_2(x,t)=u_R(x,t);$$
$$i_1(x,t)=i_G(x,t); \quad i_2(x,t)=i_R(x,t)$$
$$C_{11}=C_G+C_m,\; C_{12}=C_{21}=-C_m,\; C_{22}=C_R+C_m,$$
$$L_{11}=L_G,\; L_{12}=L_{21}=L_m,\; L_{22}=L_R$$

In the new denotations we reach the following boundary conditions

$$\left.\begin{aligned} &u_1(0,t)=U_S(t)-R_S i_1(0,t) \\ &U_{NE}=u_2(0,t)=-R_{NE} i_2(0,t) \end{aligned}\right\} \text{at the near end}$$
$$\left.\begin{aligned} &C_0 \frac{du_1(\Lambda,t)}{dt} = i_1(\Lambda,t) - f_1(u_1(\Lambda,t)) \\ &C_0 \frac{du_2(\Lambda,t)}{dt} = i_2(\Lambda,t) - f_2(u_2(\Lambda,t)) \end{aligned}\right\} \text{at the far end} \tag{5}$$

where $t \geq 0$ and initial conditions (at $t = 0$)

$$\begin{aligned} &u_1(x,0)=u_{10}(x),\; u_2(x,0)=u_{20}(x),\; x\in[0,\Lambda] \\ &i_1(x,0)=i_{10}(x),\;\; i_2(x,0)=i_{20}(x),\;\; x\in[0,\Lambda] \end{aligned} \tag{6}$$

II. Solution to the 3-conductor Transmission Line System of Equations

Finding a solution of the mixed problem for the hyperbolic system describing the 3-conductor transmission line system (with the transmission lines terminated by nonlinear resistive loads), means finding the currents and voltages along the lines. Here, the hyperbolic system is transformed to a diagonal form. Next, we reduce the mixed problem for the hyperbolic system to an initial value problem on the boundary. The obtained equations are of neutral type [14], which will be solved in a next paper.

A. Hyperbolic System Transformation

In a matrix form (4) becomes

$$A\frac{\partial U}{\partial t} + M\frac{\partial U}{\partial x} = 0 \tag{7}$$

where we used the following denotations for the matrices:

$$A=\begin{pmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22} \end{pmatrix}; M=\begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix};$$
$$\frac{\partial U}{\partial t}=\begin{bmatrix} \partial u_1/\partial t \\ \partial u_2/\partial t \\ \partial i_1/\partial t \\ \partial i_2/\partial t \end{bmatrix}; \frac{\partial U}{\partial x}=\begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \\ \partial i_1/\partial x \\ \partial i_2/\partial x \end{bmatrix} \tag{8}$$

Since $\Delta_C = C_{11}C_{22} - C_{12}^2 = (C_G + C_m)(C_R + C_m) - C_m^2 = = C_G C_R + C_G C_m + C_R C_m > 0$

we have to assume the following:

Assumption (L). The matrix L is non-singular that is its determinant is different from zero:

$$\Delta_L = L_G L_R - L_m^2 = L_{11}L_{22} - L_{12}^2 \neq 0 .$$

This implies that $|A| = \Delta_C \Delta_L \neq 0$ and therefore A^{-1} does exist.

We denote by B the following matrix: $B \equiv A^{-1}M$ and we obtain

$$\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} = 0 \tag{9}$$

Let us substitute $U(x,t) = H\,Z(x,t)$ in (9) where $Z = (I_1, I_2, I_3, I_4)^T$; I_k (k = 1,2,3,4) – new unknown functions. We obtain

$$H\frac{\partial Z(x,t)}{\partial t} + BH\frac{\partial Z(x,t)}{\partial x} = 0$$

and multiply by H^{-1} from the left to obtain

$$\frac{\partial Z(x,t)}{\partial t} + H^{-1}BH\frac{\partial Z(x,t)}{\partial x} = 0 .$$

H needs to be found so that $H^{-1}BH = B^{can}$, where

$$B^{can} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

and λ_k (k=1,2,3,4) are the eigen values of B, hence the roots of

$$|B-\lambda I| = \\ = \begin{vmatrix} -\lambda & 0 & C_{22}/\Delta_C & -C_{12}/\Delta_C \\ 0 & -\lambda & -C_{12}/\Delta_C & C_{11}/\Delta_C \\ L_{22}/\Delta_L & -L_{12}/\Delta_L & -\lambda & 0 \\ -L_{12}/\Delta_L & L_{11}/\Delta_L & 0 & -\lambda \end{vmatrix} = \\ = \frac{\Delta_C\Delta_L\lambda^4 - (L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})\lambda^2 + 1}{\Delta_C\Delta_L} = 0 \tag{10}$$

where I – identity matrix, $|B - \lambda I|$ – determinant of $B - \lambda I$. We suppose the following assumption:

Assumption (D). The discriminant D of the above bi-quadratic equation in (10) is positive:

$$D = (L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})^2 - \\ -4\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right) > 0$$

We find characteristic roots from (10)

$$\lambda_1 = \sqrt{\frac{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)} + \frac{\sqrt{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})^2 - 4\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}$$

$$\lambda_2 = \sqrt{\frac{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)} - \frac{\sqrt{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})^2 - 4\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}$$

$$\lambda_3 = -\sqrt{\frac{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)} + \frac{\sqrt{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})^2 - 4\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}$$

$$\lambda_4 = -\sqrt{\frac{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)} - \frac{\sqrt{(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22})^2 - 4\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}$$

For simplicity, we the find the eigen-vectors of

$$\left(B^{-1} - \mu_k I\right)H^{(k)} = 0;\ \mu_k = 1/\lambda_k;\ H^{(k)} = (\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k})^T$$

(instead of $(B - \lambda_k I)H^{(k)} = 0$) because

$$B^{-1} = \begin{pmatrix} 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22} \\ C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \end{pmatrix}$$

has a simpler form than B.

With the above roots we obtain 4 eigen-vectors $H^{(k)}$ (k=1,2,3,4).

$$\begin{aligned} (B - \lambda_1 I)H^{(1)} = 0 \quad &; \quad (B - \lambda_2 I)H^{(2)} = 0 \\ (B - \lambda_3 I)H^{(3)} = 0 \quad &; \quad (B - \lambda_4 I)H^{(4)} = 0 \end{aligned} \qquad (11)$$

denoted by $H^{(k)} = (\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k})^T; (k = 1,2,3,4)$.

To solve (11) we have to assume:

$$L_{12}C_{11} + L_{22}C_{12} = L_m(C_G + C_m) - L_R C_m \neq 0$$
$$L_{12}C_{22} + L_{11}C_{12} = L_m(C_R + C_m) - L_G C_m \neq 0.$$

Note that the characteristic roots of (10) satisfy the following inequalities $\lambda_1 > \lambda_2 > 0$; $\lambda_3 = -\lambda_1$; $\lambda_4 = -\lambda_2$.

We obtain the following eigen-vectors

$$H^{(1)} = (p_1, q_1, 1, \gamma_1)^T$$

$$H^{(2)} = (p_2, q_2, 1, \gamma_2)^T$$

$$H^{(3)} = (-p_1, -q_1, 1, \gamma_1)^T$$

$$H^{(4)} = (-p_2, -q_2, 1, \gamma_2)^T$$

Then the transformation matrix becomes

$$H = \begin{bmatrix} p_1 & p_2 & -p_1 & -p_2 \\ q_1 & q_2 & -q_1 & -q_2 \\ 1 & 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \gamma_1 & \gamma_2 \end{bmatrix}$$

where

$$p_k = \frac{L_{12} + \lambda_k^2 \Delta_L C_{12}}{\lambda_k (L_{12}C_{11} + L_{22}C_{12})} = \lambda_k (L_{11} + L_{12}\gamma_k)$$

$$q_k = \frac{L_{22} - \lambda_k^2 \Delta_L C_{11}}{\lambda_k (L_{12}C_{11} + L_{22}C_{12})} = \lambda_k (L_{12} + L_{22}\gamma_k)$$

$$\gamma_k = \frac{1 - \lambda_k^2 (L_{11}C_{11} + L_{12}C_{12})}{\lambda_k^2 (L_{12}C_{11} + L_{22}C_{12})}$$

$(k = 1,2)$

Since $|H| = 4\sqrt{\Delta_L / \Delta_C}\,(\gamma_2 - \gamma_1)^2 \neq 0$ for the inverse matrix we obtain:

$$H^{-1} = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_2\sqrt{\Delta_L/\Delta_C} & -p_2\sqrt{\Delta_L/\Delta_C} & \gamma_2 & -1 \\ -q_1\sqrt{\Delta_L/\Delta_C} & p_1\sqrt{\Delta_L/\Delta_C} & -\gamma_1 & 1 \\ -q_2\sqrt{\Delta_L/\Delta_C} & p_2\sqrt{\Delta_L/\Delta_C} & \gamma_2 & -1 \\ q_1\sqrt{\Delta_L/\Delta_C} & -p_1\sqrt{\Delta_L/\Delta_C} & -\gamma_1 & 1 \end{bmatrix}.$$

B. Boundary Conditions with Respect to the New Variables

Introduce new variables $U = HZ$ and $Z = H^{-1}U$, where

$$U = (u_1, u_2, i_1, i_2)^T, \ Z = (I_1, I_2, I_3, I_4)^T$$

Then

$$u_1(x,t) = p_1 I_1(x,t) + p_2 I_2(x,t) - p_1 I_3(x,t) - p_2 I_4(x,t)$$
$$u_2(x,t) = q_1 I_1(x,t) + q_2 I_2(x,t) - q_1 I_3(x,t) - q_2 I_4(x,t)$$

$$i_1(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t)$$
$$i_2(x,t) = \gamma_1 I_1(x,t) + \gamma_2 I_2(x,t) + \gamma_1 I_1(x,t) + \gamma_2 I_2(x,t)$$

and

$$I_1(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_2 \sqrt{\Delta_C / \Delta_L}\, u_1(x,t) - \\ -p_2 \sqrt{\Delta_C / \Delta_L}\, u_2(x,t) + \\ +\gamma_2 i_1(x,t) - i_2(x,t) \end{pmatrix}$$

$$I_2(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} -q_1 \sqrt{\Delta_C / \Delta_L}\, u_1(x,t) + \\ +p_1 \sqrt{\Delta_C / \Delta_L}\, u_2(x,t) - \\ -\gamma_1 i_1(x,t) + i_2(x,t) \end{pmatrix}$$

$$I_3(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} -q_2 \sqrt{\Delta_C / \Delta_L}\, u_1(x,t) + \\ +p_2 \sqrt{\Delta_C / \Delta_L}\, u_2(x,t) + \\ +\gamma_2 i_1(x,t) - i_2(x,t) \end{pmatrix}$$

$$I_4(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_1 \sqrt{\Delta_C / \Delta_L}\, u_1(x,t) - \\ -p_1 \sqrt{\Delta_C / \Delta_L}\, u_2(x,t) - \\ -\gamma_1 i_1(x,t) + i_2(x,t) \end{pmatrix}$$

So, the mixed problem (1)-(4) is transformed to find a solution of the diagonal system:

$$\begin{aligned} &\frac{\partial I_1(x,t)}{\partial t} + \lambda_1 \frac{\partial I_1(x,t)}{\partial x} = 0, \quad \frac{\partial I_2(x,t)}{\partial t} + \lambda_2 \frac{\partial I_2(x,t)}{\partial x} = 0, \\ &\frac{\partial I_3(x,t)}{\partial t} - \lambda_1 \frac{\partial I_3(x,t)}{\partial x} = 0, \quad \frac{\partial I_4(x,t)}{\partial t} - \lambda_2 \frac{\partial I_4(x,t)}{\partial x} = 0 \end{aligned} \tag{12}$$

with the following initial conditions rewritten in the new variables

$$I_1(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_2 \sqrt{\Delta_C / \Delta_L}\, u_{10}(x) - \\ -p_2 \sqrt{\Delta_C / \Delta_L}\, u_{20}(x) + \\ +\gamma_2 i_{10}(x) - i_{20}(x) \end{pmatrix} \equiv I_{10}(x)$$

$$I_2(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} -q_1 \sqrt{\Delta_C / \Delta_L}\, u_{10}(x) + \\ +p_1 \sqrt{\Delta_C / \Delta_L}\, u_{20}(x) - \\ -\gamma_1 i_{10}(x) + i_{20}(x) \end{pmatrix} \equiv I_{20}(x)$$

(13)

$$I_3(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_2 \sqrt{\Delta_C / \Delta_L}\, u_{10}(x) + \\ +p_2 \sqrt{\Delta_C / \Delta_L}\, u_{20}(x) + \\ +\gamma_2 i_{10}(x - i_{20}(x) \end{pmatrix} \equiv I_{30}(x)$$

$$I_4(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_1 \sqrt{\Delta_C / \Delta_L}\, u_{10}(x) - \\ -p_1 \sqrt{\Delta_C / \Delta_L}\, u_{20}(x) - \\ -\gamma_1 i_{10}(x) + i_{20}(x) \end{pmatrix} \equiv I_{40}(x)$$

and boundary conditions rewritten in the new variables (at $t \geq 0$):

$$\begin{aligned} &p_1 I_1(0,t) + p_2 I_2(0,t) - p_1 I_3(0,t) - p_2 I_4(0,t) = \\ &\quad = U_S(t) - R_S \left[I_1(0,t) + I_2(0,t) + I_3(0,t) + I_4(0,t) \right] \end{aligned}$$

$$\begin{aligned} &q_1 I_1(0,t) + q_2 I_2(0,t) - q_1 I_3(0,t) - q_2 I_4(0,t) = \\ &\quad = -R_{NE} \left[\gamma_1 I_1(0,t) + \gamma_2 I_2(0,t) + \gamma_1 I_3(0,t) + \gamma_2 I_4(0,t) \right] \end{aligned} \tag{14}$$

$$\begin{aligned} &C_1 \frac{d\left(p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t) \right)}{dt} = \\ &\quad = I_1(\Lambda,t) + I_2(\Lambda,t) + I_3(\Lambda,t) + I_4(\Lambda,t) - \\ &\quad - f_1 \begin{pmatrix} p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - \\ -p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &C_2 \frac{d\left(q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t) \right)}{dt} = \\ &\quad = \gamma_1 I_1(\Lambda,t) + \gamma_2 I_2(\Lambda,t) + \gamma_1 I_3(\Lambda,t) + \gamma_2 I_4(\Lambda,t) - \\ &\quad - f_2 \begin{pmatrix} q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - \\ -q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t) \end{pmatrix} \end{aligned}$$

C. Derivation of a Neutral System Equivalent to the Mixed Problem

Proceeding as in [13] we find the characteristics of the system (12) which forms 4 families of curves

$$\frac{dx}{dt} = \lambda_1 ; \frac{dx}{dt} = \lambda_2 ; \frac{dx}{dt} = -\lambda_1 ; \frac{dx}{dt} = -\lambda_2 \tag{15}$$

Through each point $(x,t) \in \Pi = \{(x,t) \in [0,\Lambda] \times [0,T]\}$ there are 4 characteristics: C_1, C_2 with positive slopes and C_3, C_4, with negative slopes. A characteristic C_k (k=1,2) through a point $(0,\hat{t}_k)$ intersects the boundary $x = \Lambda$ at some point $(\Lambda, \hat{t}_k + T_k)$ where T_k can be found by integration of $\frac{dx}{dt} = \lambda_k$ Since the characteristic C_k is $x - \lambda_k t = \text{const}$, then the straight line through $(0,\hat{t}_k)$ is

$$x - \lambda_k t = -\lambda_k \hat{t}_k \Rightarrow t = \frac{x}{\lambda_k} + \hat{t}_k$$

Setting $x = \Lambda$ and $t = \hat{t}_k + T_k$ we obtain

$$\Lambda - \lambda_k \left(\hat{t} + T_k \right) = -\lambda_k \hat{t} \;\Rightarrow\; T_k = \Lambda / \lambda_k .$$

Similarly, the characteristic $C_p \, (p = 3,4)$ is $x + \lambda_p t = \text{const}$ (with $\lambda_3 = -\lambda_1, \lambda_4 = -\lambda_2$) and the straight line through a point $(\Lambda, \hat{t}_p)$ is $x + \lambda_p t = \Lambda + \lambda_p \hat{t}_p$. It intersects x = 0 at a point $(0, \hat{t}_p + T_p)$. Therefore

$$\lambda_p \left(\hat{t}_p + T_p \right) = \Lambda + \lambda_p \hat{t}_p \Rightarrow T_p = \Lambda / \lambda_p \; (p = 3,4)$$

i.e. $T_3 = T_1$, $T_4 = T_2$.

Introduce directional derivatives

$$D_k = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{d}{dt} = \frac{\partial}{\partial t} + \lambda_k \frac{\partial}{\partial x}(k = 1,2)$$

$$D_k = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{d}{dt} = \frac{\partial}{\partial t} - \lambda_k \frac{\partial}{\partial x}(k = 3,4)$$

Then system (12) can be written in the form:

$$D_k I_k = 0 \ (k = 1,2,3,4) \quad (16)$$

Integrating the equations from (16) for k=1,2 along the characteristic C_k from $(0, t)$ to $(\Lambda, t + T_k)$ (where the integration is a line integral along C_k) we obtain

$$I_k(\Lambda, t + T_k) = I_k(0,t) \ (t \geq 0)$$

Similarly, the equation from (16) for k=3,4 along C_k from $(0, t + T_k)$ to (Λ, t) we get

$$I_k(\Lambda, t) = I_k(0, t + T_k) \ (t \geq 0)$$

From (14) the initial conditions imply:

$$(p_1 + R_S) I_1(0,t) + (p_2 + R_S) I_2(0,t) = \\ = U_S(t) + (p_1 - R_S) I_3(0,t) + (p_2 - R_S) I_4(0,t) \quad (17)$$

$$(q_1 + R_{NE}\gamma_1) I_1(0,t) + (q_2 + R_{NE}\gamma_2) I_2(0,t) = \\ = (q_1 - R_{NE}\gamma_1) I_3(0,t) + (q_2 - R_{NE}\gamma_2) I_4(0,t)$$

while boundary conditions imply:

$$p_1 \frac{dI_3(\Lambda,t)}{dt} + p_2 \frac{dI_4(\Lambda,t)}{dt} = \\ = p_1 \frac{dI_1(\Lambda,t)}{dt} + p_2 \frac{dI_2(\Lambda,t)}{dt} - \\ - \frac{I_1(\Lambda,t) + I_2(\Lambda,t) + I_3(\Lambda,t) + I_4(\Lambda,t)}{C_1} + \\ + \frac{f_1(p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t))}{C_1}, \quad (18)$$

$$q_1 \frac{dI_3(\Lambda,t)}{dt} + q_2 \frac{dI_4(\Lambda,t)}{dt} = \\ = q_1 \frac{dI_1(\Lambda,t)}{dt} + q_2 \frac{dI_2(\Lambda,t)}{dt} - \\ - \frac{\gamma_1 I_1(\Lambda,t) + \gamma_2 I_2(\Lambda,t) + \gamma_1 I_3(\Lambda,t) + \gamma_2 I_4(\Lambda,t)}{C_2} + \\ + \frac{f_2(q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t))}{C_2}$$

Solving (17) with respect to $I_{1\ sss}(0, t)$ and $I_2(0, t)$ we obtain

$$I_1(0,t) = A_{10}(t) + A_{11} I_3(0,t) + A_{12} I_4(0,t)$$
$$I_2(0,t) = A_{20}(t) + A_{21} I_3(0,t) + A_{22} I_4(0,t)$$

where the coefficients A_{10}, A_{11}, A_{12}, A_{20}, A_{21}, A_{22} are as follows:

$$A_{10}(t) = \frac{(q_2 + R_{NE}\gamma_2) U_S(t)}{\Delta_{12}};$$

$$A_{11} = \frac{2(p_2\gamma_2 R_{NE} - q_2 R_S)}{\Delta_{12}};$$

$$A_{12} = \frac{p_1 q_2 - p_2 q_1 - (q_2 + q_1) R_S}{\Delta_{12}} + \\ + \frac{(p_1\gamma_2 + p_2\gamma_1) R_{NE} + (\gamma_1 - \gamma_2) R_S R_{NE}}{\Delta_{12}}$$

$$A_{20}(t) = -\frac{(q_1 + R_{NE}\gamma_1) U_S(t)}{\Delta_{12}};$$

$$A_{21} = \frac{p_1 q_2 - p_2 q_1 + (q_2 + q_1) R_S}{\Delta_{12}} - \\ - \frac{(p_1\gamma_2 + p_2\gamma_1) R_{NE} + (\gamma_1 - \gamma_2) R_S R_{NE}}{\Delta_{12}};$$

$$A_{22} = \frac{-2 p_1 \gamma_1 + 2 q_1 R_S}{\Delta_{12}}$$

Solving (18) with respect to $\frac{dI_3(\Lambda,t)}{dt}$ and $\frac{dI_4(\Lambda,t)}{dt}$ we obtain

$$\frac{dI_3(\Lambda,t)}{dt} = \frac{dI_1(\Lambda,t)}{dt} + \\ + \frac{p_2\gamma_1 - q_2}{\Delta_{34} C_0} I_1(\Lambda,t) + \frac{p_2\gamma_2 - q_2}{\Delta_{34} C_0} I_2(\Lambda,t) + \\ + \frac{p_2\gamma_1 - q_2}{\Delta_{34} C_0} I_3(\Lambda,t) + \frac{p_2\gamma_2 - q_2}{\Delta_{34} C_0} I_4(\Lambda,t) + \\ + \frac{q_2}{\Delta_{34} C_0} f_1(p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t)) - \\ - \frac{p_2}{\Delta_{34} C_0} f_2(q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t))$$

$$\frac{dI_4(\Lambda,t)}{dt} = \frac{dI_2(\Lambda,t)}{dt} - \\ - \frac{p_1\gamma_1 + q_1}{\Delta_{34} C_0} I_1(\Lambda,t) - \frac{p_1\gamma_2 + q_1}{\Delta_{34} C_0} I_2(\Lambda,t) - \\ - \frac{p_1\gamma_1 + q_1}{\Delta_{34} C_0} I_3(\Lambda,t) - \frac{p_1\gamma_2 + q_1}{\Delta_{34} C_0} I_4(\Lambda,t) \\ + \frac{p_1}{\Delta_{34} C_0} f_2(q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t)) + \\ + \frac{q_1}{\Delta_{34} C_0} f_1(p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t))$$

Considering the relations

$$I_1(\Lambda,t) = I_1(0, t - T_1), I_2(\Lambda,t) = I_2(0, t - T_2)$$

$$I_3(\Lambda, t - T_3) = I_3(0,t), I_4(\Lambda, t - T_4) = I_4(0,t)$$

and designating the unknown functions by

$$I_1(0,t) \equiv I_1(t)$$
$$I_2(0,t) \equiv I_2(t)$$
$$I_3(t) \equiv I_3(\Lambda,t)$$
$$I_4(t) \equiv I_4(\Lambda,t)$$

and taking into account $T_1 = T_3, T_2 = T_4$, we reach the system

$$I_1(t) = A_{10}(t) + A_{11}I_3(t-T_1) + A_{12}I_4(t-T_2) \equiv U_1 \quad (19)$$

$$I_2(t) = A_{20}(t) + A_{21}I_3(t-T_1) + A_{22}I_4(t-T_2) \equiv U_2 \quad (20)$$

$$\begin{aligned}\frac{dI_3(t)}{dt} = & \frac{dI_1(t-T_1)}{dt} + \\ & + \frac{p_2\gamma_1 - q_2}{\Delta_{34}C_0} I_1(t-T_1) + \\ & + \frac{p_2\gamma_2 - q_2}{\Delta_{34}C_0} I_2(t-T_2) + \\ & + \frac{p_2\gamma_1 - q_2}{\Delta_{34}C_0} I_3(t) + \\ & + \frac{p_2\gamma_2 - q_2}{\Delta_{34}C_0} I_4(t) + \\ & + \frac{q_2}{\Delta_{34}C_0} f_1\begin{pmatrix} p_1 I_1(t-T_1) + p_2 I_2(t-T_2) - \\ -p_1 I_3(t) - p_2 I_4(t) \end{pmatrix} - \\ & - \frac{p_2}{\Delta_{34}C_0} f_2\begin{pmatrix} q_1 I_1(t-T_1) + q_2 I_2(t-T_2) - \\ -q_1 I_3(t) - q_2 I_4(t) \end{pmatrix} \equiv \\ & \equiv U_3 \end{aligned} \quad (21)$$

$$\begin{aligned}\frac{dI_4(t)}{dt} = & \frac{dI_2(t-T_2)}{dt} - \\ & - \frac{p_1\gamma_1 + q_1}{\Delta_{34}C_0} I_1(t-T_1) - \\ & - \frac{p_1\gamma_2 + q_1}{\Delta_{34}C_0} I_2(t-T_2) - \\ & - \frac{p_1\gamma_1 + q_1}{\Delta_{34}C_0} I_3(t) - \\ & - \frac{p_1\gamma_2 + q_1}{\Delta_{34}C_0} I_4(t) + \\ & + \frac{p_1}{\Delta_{34}C_0} f_2\begin{pmatrix} q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - \\ -q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t) \end{pmatrix} + \\ & + \frac{q_1}{\Delta_{34}C_0} f_1\begin{pmatrix} p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - \\ -p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t) \end{pmatrix} \equiv \\ & \equiv U_4 \end{aligned} \quad (22)$$

To obtain the initial conditions on the intervals $[-T_1,0]$, $[-T_2,0]$, we could shift the initial functions

$$u_{10}(x), u_{20}(x), i_{10}(x), i_{20}(x)$$

from the interval $[0,\Lambda]$ along the characteristics to intervals $[-T_1,0]$, $[-T_2,0]$.

The resultant functions after the above transformation on the boundary we denote by

$$I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)$$

If functions $u_{10}(x), u_{20}(x), i_{10}(x), i_{20}(x)$ are periodic, functions $I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)$ are also periodic.

III. Conclusion

We have reduced the mixed problem for 3-conductor transmission line to an initial value problem on the boundary for the system (19)-(22). It consists of two functional equations (19) and (20) and two neutral equations (21) and (22) with two different delays for the unknown functions I_1, I_2, I_3, I_4. By applying a fixed-point method this problem can be solved at the given initial conditions, which will be done in a next paper.

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