Note on an Generalized Solution of the Three-Conductor Transmission Line Equations

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***Abstract*— Here we present an abridged investigation on the electromagnetic compatibility aspects of three lossless transmission lines terminated by linear loads introduced by C. Paul. Taking into account the mutual interaction between two lines we do not neglect the influence of the receptor line i.e. we do not apply the weak-coupling approximation. This leads to a more general mathematical model than the model of C. Paul. We formulate a mixed problem for the hyperbolic system describing the three-conductor transmission line. It is proved that the mixed problem is equivalent to an initial value problem for a functional system on the boundary of hyperbolic system's domain. The unknown functions in this system are the lines' voltages and currents. The obtained system of functional equations can be solved by a fixed-point method that enables us to find an approximated but explicit solution. Usually such problems are treated by numerical methods or Laplace transformation method, which are applicable to linear problems only.**

Keywords*— *electromagnetic compatibility, 3-conductor transmission line, linear hyperbolic system, initial-boundary (mixed) problem for hyperbolic system, fixed point method

I. INTRODUCTION

Many studies have been devoted to the investigation of VLSI systems and their applications (cf. [1]-[8]). Here we consider an electromagnetic compatibility model of a three-conductor transmission line using the results of C. Paul [9]. Unlike [9], we propose a general approach to solve the problem in question and show that the weak coupling assumptions introduced in [9] (cf. also [10]) turns out to be a particular case of our more general treatment.

Following the technique from [11]-[13] (applied also to other problems, e.g. [14], [15]) we obtain a general solution of the system modelling pairwise interacting 3-conductor transmission line introduced in [9]. We proceed from the 3-conductor transmission line circuit shown in Fig. 1 (cf. [9]). The ground symbol in Fig. 1 denotes the reference conductor for the line voltages. In our case, this is a PCB land although it may also be an infinite ground plane, a wire, an overall shield, etc. The other two conductors are also PCB lands though in general they may be of various type too. The line is assumed to be an uniform and lossless line (cf. [7], [8]).

The upper circuit is referred to as a generator circuit; it is driven by a voltage source with open-circuit voltage $U_S(t)$ and source resistance R_S; it is terminated by a resistive load R_L .The lower circuit is referred to as a receptor circuit; it is terminated by a resistive load R_{NE} at the near end and by a resistive load R_{FE} at the far end. Electric and magnetic fields, arisen by the voltage and current of the generator circuit, interact with the receptor circuit producing crosstalk voltages at the terminals of the receptor circuit.

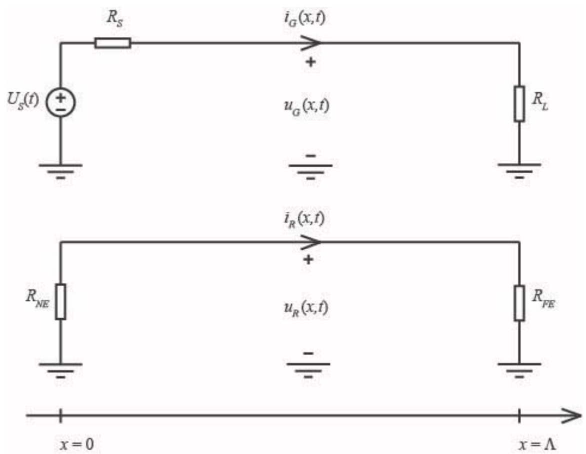

Fig. 1. Circuit of the 3-conductor transmission line.

Our objective is to find a solution for these crosstalk voltages proceeding from a more general system compared to the one in [9], that is the hyperbolic system (1) obtained in accordance to the TEM mode of propagation (cf. [1]-[8]). The voltages with respect to the reference conductor u_k (x, t) $(k = 1; 2)$ and the currents of each circuit i_k (x, t) $(k = 1, 2)$ are functions of position x and time t.

$$\left|\begin{array}{l}\dfrac{\partial u_G(x,t)}{\partial x}+L_G\dfrac{\partial i_G(x,t)}{\partial t}=-L_m\dfrac{\partial i_R(x,t)}{\partial t}\\ \dfrac{\partial i_G(x,t)}{\partial x}+(C_G+C_m)\dfrac{\partial u_G(x,t)}{\partial t}=C_m\dfrac{\partial u_R(x,t)}{\partial t}\\ \dfrac{\partial u_R(x,t)}{\partial x}+L_R\dfrac{\partial i_R(x,t)}{\partial t}=-L_m\dfrac{\partial i_G(x,t)}{\partial t}\\ \dfrac{\partial i_R(x,t)}{\partial x}+(C_R+C_m)\dfrac{\partial u_R(x,t)}{\partial t}=C_m\dfrac{\partial u_G(x,t)}{\partial t}\end{array}\right. \tag{1}$$

with the following boundary

$$\begin{array}{ll}u_G(0,t)=U_S(t)-R_Si_G(0,t), & U_{NE}=u_R(0,t)=-R_{NE}i_R(0,t)\\ u_G(\Lambda,t)=R_\Lambda i_G(\Lambda,t), & U_{FE}=u_R(\Lambda,t)=R_{FE}i_R(\Lambda,t)\end{array} \tag{2}$$

and initial conditions:

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$$u_G(x,0)=u_{G0}(x), \qquad u_R(x,0)=u_{R0}(x)$$
$$i_G(x,0)=i_{G0}(x), \qquad i_R(x,0)=i_{R0}(x),\ x\in[0,\Lambda] \tag{3}$$

Before continuing we point out that in our investigation, we do not apply the weak coupling assumption as distinct from [9] where weak coupling assumption is applied; this means that the right-hand side of (1) is not neglected. Hence, our method enables us consider the more general case of (1).

Introduce denotations

$$u_1(x,t)=u_G(x,t); \quad u_2(x,t)=u_R(x,t);$$
$$i_1(x,t)=i_G(x,t); \quad i_2(x,t)=i_R(x,t)$$

$$C_{11}=C_G+C_m,\ C_{12}=C_{21}=-C_m,\ C_{22}=C_R+C_m$$
$$L_{11}=L_G,\ L_{12}=L_{21}=L_m,\ L_{22}=L_R$$

$$C=\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22}\end{pmatrix},\ L=\begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22}\end{pmatrix}$$

We are able to formulate the following mixed problem: to find a solution of the system

$$\begin{aligned}
&C_{11}\frac{\partial u_1(x,t)}{\partial t}+C_{12}\frac{\partial u_2(x,t)}{\partial t}+\frac{\partial i_1(x,t)}{\partial x}=0,\\
&C_{12}\frac{\partial u_1(x,t)}{\partial t}+C_{22}\frac{\partial u_2(x,t)}{\partial t}+\frac{\partial i_2(x,t)}{\partial x}=0,\\
&L_{11}\frac{\partial i_1(x,t)}{\partial t}+L_{12}\frac{\partial i_2(x,t)}{\partial t}+\frac{\partial u_1(x,t)}{\partial x}=0,\\
&L_{12}\frac{\partial i_1(x,t)}{\partial t}+L_{22}\frac{\partial i_2(x,t)}{\partial t}+\frac{\partial u_2(x,t)}{\partial x}=0
\end{aligned},$$

where

$$u_1(0,t)=U_S(t)-R_Si_1(0,t), \qquad U_{NE}=u_2(0,t)=-R_{NE}i_2(0,t),$$
$$u_1(\Lambda,t)=R_Li_1(\Lambda,t), \qquad U_{FE}=u_2(\Lambda,t)=R_{FE}i_2(\Lambda,t) \tag{4}$$

$$u_1(x,0)=u_{10}(x), \qquad u_2(x,0)=u_{20}(x),\ x\in[0,\Lambda]$$
$$i_1(x,0)=i_{10}(x), \qquad i_2(x,0)=i_{20}(x),\ x\in[0,\Lambda]$$

II. TRANSFORMATION OF THE HYPERBOLIC SYSTEM AND RESULTS

In a matrix form the above system (4) is

$$\begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22}\end{bmatrix}\begin{bmatrix}\partial u_1/\partial t \\ \partial u_2/\partial t \\ \partial i_1/\partial t \\ \partial i_2/\partial t\end{bmatrix}+ \\ +\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\partial u_1/\partial x \\ \partial u_2/\partial x \\ \partial i_1/\partial x \\ \partial i_2/\partial x\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{5}$$

Since $\Delta_C=C_{11}C_{22}-C_{12}^2=C_GC_R+C_GC_m+C_RC_m>0$ and **Assumption (L)**: $\Delta_L=L_GL_R-L_m^2=L_{11}L_{22}-L_{12}^2\neq 0$

we obtain $|A|=\begin{vmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22}\end{vmatrix}=\Delta_C\Delta_L\neq 0$ and

therefore A^{-1} does exist and we have

$$\begin{bmatrix}\frac{\partial u_1}{\partial t}\\ \frac{\partial u_2}{\partial t}\\ \frac{\partial i_1}{\partial t}\\ \frac{\partial i_2}{\partial t}\end{bmatrix}+\begin{bmatrix} 0 & 0 & \frac{C_{22}}{\Delta_C} & \frac{-C_{12}}{\Delta_C} \\ 0 & 0 & \frac{-C_{12}}{\Delta_C} & \frac{C_{11}}{\Delta_C} \\ \frac{L_{22}}{\Delta_L} & \frac{-L_{12}}{\Delta_L} & 0 & 0 \\ \frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial u_1}{\partial x}\\ \frac{\partial u_2}{\partial x}\\ \frac{\partial i_1}{\partial x}\\ \frac{\partial i_2}{\partial x}\end{bmatrix}=0 \tag{6}$$

Rewrite (6) in a matrix form

$$\frac{\partial U(x,t)}{\partial t}+B\frac{\partial U(x,t)}{\partial x}=0 \tag{7}$$

Substitute $U(x,t)=H\,Z(x,t)$ in (7):

$$H\frac{\partial Z(x,t)}{\partial t}+BH\frac{\partial Z(x,t)}{\partial x}=0$$

and multiplying by H^{-1} we obtain

$$\frac{\partial Z(x,t)}{\partial t}+H^{-1}BH\frac{\partial Z(x,t)}{\partial x}=0$$

We have to find H such that $H^{-1}BH=B^{\mathrm{can}}$, where B^{can} is a diagonal matrix whose elements are eigen-values $\lambda_k(k=1,2,3,4)$ of B, i.e. the roots of the characteristic equation

$$|B-\lambda I|= \\ =\frac{\Delta_C\Delta_L\lambda^4-\left(L_{11}C_{11}+2L_{12}C_{12}+L_{22}C_{22}\right)\lambda^2+1}{\Delta_C\Delta_L}=0$$

Under **Assumption (D):**

$$D=\left(L_{11}C_{11}+2L_{12}C_{12}+L_{22}C_{22}\right)^2- \\ -4\left(C_{11}C_{22}-C_{12}^2\right)\left(L_{11}L_{22}-L_{12}^2\right)>0$$

One can obtain characteristic roots and then we find corresponding eigenvector

$$H^{(k)}=\left(\xi_{1k},\xi_{2k},\xi_{3k},\xi_{4k}\right)^T; \quad (k=1,2,3,4).$$

$$\gamma_k = \frac{1-\lambda_k^2\left(L_{11}C_{11}+L_{12}C_{12}\right)}{\lambda_k^2\left(L_{12}C_{11}+L_{22}C_{12}\right)} \quad (k=1,2).$$

The transformation matrix is

$$H = \begin{bmatrix} p_1 & p_2 & -p_1 & -p_2 \\ q_1 & q_2 & -q_1 & -q_2 \\ 1 & 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \gamma_1 & \gamma_2 \end{bmatrix}$$

$$|H| = 4\sqrt{\Delta_L/\Delta_C}\left(\gamma_2-\gamma_1\right)^2 \neq 0 .$$

III. Derivation of the Boundary Conditions with Respect to the New Variables

The transformation formulas between

$U = \left(u_1, u_2, i_1, i_2\right)^T$ and $Z = \left(I_1, I_2, I_3, I_4\right)^T$ are

$$u_1(x,t) = p_1 I_1(x,t) + p_2 I_2(x,t) - p_1 I_3(x,t) - p_2 I_4(x,t)$$

$$u_2(x,t) = q_1 I_1(x,t) + q_2 I_2(x,t) - q_1 I_3(x,t) - q_2 I_4(x,t)$$

$$i_1(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t)$$

$$i_2(x,t) = \gamma_1 I_1(x,t) + \gamma_2 I_2(x,t) + \gamma_1 I_3(x,t) + \gamma_2 I_4(x,t)$$

With respect to the new variables the mixed problem (1)-(3) becomes as follows: to find a solution of the system

$$\begin{aligned} &\frac{\partial I_1(x,t)}{\partial t} + \lambda_1 \frac{\partial I_1(x,t)}{\partial x} = 0, \quad \frac{\partial I_2(x,t)}{\partial t} + \lambda_2 \frac{\partial I_2(x,t)}{\partial x} = 0, \\ &\frac{\partial I_3(x,t)}{\partial t} - \lambda_1 \frac{\partial I_3(x,t)}{\partial x} = 0, \quad \frac{\partial I_4(x,t)}{\partial t} - \lambda_2 \frac{\partial I_4(x,t)}{\partial x} = 0 \end{aligned} \tag{8}$$

with initial conditions and boundary conditions in the new variables: □

$$I_1(x,0) = I_{10}(x), I_2(x,0) = I_{20}(x), I_3(x,0) = I_{30}(x), I_4(x,0) = I_{40}(x)$$

To obtain the boundary conditions

$$\begin{aligned} &u_1(0,t) = U_S(t) - R_S i_1(0,t), \quad u_1(\Lambda,t) = R_\Lambda i_1(\Lambda,t), \\ &U_{NE} = u_2(0,t) = -R_{NE} i_2(0,t), \quad U_{FE} = u_2(\Lambda,t) = R_{FE} i_2(\Lambda,t) \end{aligned}$$

with respect to the new variables we use the transformation formulas and obtain

$$\begin{aligned} &p_1 I_1(0,t) + p_2 I_2(0,t) - p_1 I_3(0,t) - p_2 I_4(0,t) = \\ &\quad = U_S(t) - R_S\left[I_1(0,t) + I_2(0,t) + I_3(0,t) + I_4(0,t)\right] \\ &q_1 I_1(0,t) + q_2 I_2(0,t) - q_1 I_3(0,t) - q_2 I_4(0,t) = \\ &\quad = -R_{NE}\left[\gamma_1 I_1(0,t) + \gamma_2 I_2(0,t) + \gamma_1 I_3(0,t) + \gamma_2 I_4(0,t)\right] \\ &p_1 I_1(\Lambda,t) + p_2 I_2(\Lambda,t) - p_1 I_3(\Lambda,t) - p_2 I_4(\Lambda,t) = \\ &\quad = R_\Lambda\left[I_1(\Lambda,t) + I_2(\Lambda,t) + I_3(\Lambda,t) + I_4(\Lambda,t)\right] \\ &q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - q_1 I_3(\Lambda,t) - q_2 I_4(\Lambda,t) = \\ &\quad = R_{FE}\left[\gamma_1 I_1(\Lambda,t) + \gamma_2 I_2(\Lambda,t) + \gamma_1 I_3(\Lambda,t) + \gamma_2 I_4(\Lambda,t)\right] \end{aligned} .$$

Through each point $(x,t) \in \Pi = \{(x,t) \in [0,\Lambda] \times [0,T]\}$ there are 4 characteristics: C_1, C_2 with positive slopes and C_3, C_4 with negative slopes. These are the solutions of

$$\frac{dx}{dt} = \lambda_1, \quad \frac{dx}{dt} = \lambda_2, \quad \frac{dx}{dt} = -\lambda_1, \quad \frac{dx}{dt} = -\lambda_2$$

Integrating the following equations

$$D_k I_k = 0 \; (k=1,2) \; □$$
$$D_k I_k = 0 \; (k=3,4) \; □$$

we obtain

$I_k(\Lambda, t+T_k) = I_k(0,t) \; (t \geq 0)$ and

$I_k(\Lambda,t) = I_k(0, t+T_k) \; (t \geq 0)$.

Then the boundary conditions become:

$$\begin{aligned} &\left(p_1 + R_S\right) I_1(0,t) + \left(p_2 + R_S\right) I_2(0,t) = \\ &\quad = U_S(t) + \left(p_2 - R_S\right) I_3(0,t) + \left(p_1 - R_S\right) I_4(0,t) \\ &\left(q_1 + R_{NE}\gamma_1\right) I_1(0,t) + \left(q_2 + \gamma_2 R_{NE}\right) I_2(0,t) = \\ &\quad = \left(q_2 - R_{NE}\gamma_2\right) I_3(0,t) + \left(q_1 - R_{NE}\gamma_1\right) I_4(0,t) \\ &\left(R_\Lambda + p_2\right) I_3(\Lambda,t) + \left(p_1 + R_L\right) I_4(\Lambda,t) = \\ &\quad = \left(p_1 - R_\Lambda\right) I_1(\Lambda,t) + \left(p_2 - R_L\right) I_2(\Lambda,t) \\ &\left(q_2 + R_{FE}\gamma_2\right) I_3(\Lambda,t) + \left(q_1 + R_{FE}\gamma_1\right) I_4(\Lambda,t) = \\ &\quad = \left(q_1 - R_{FE}\gamma_1\right) I_1(\Lambda,t) + \left(q_2 - R_{FE}\gamma_2\right) I_2(\Lambda,t) \end{aligned}$$

Since

$$\Delta_{12} = \begin{vmatrix} p_1 + R_S & p_2 + R_S \\ q_1 + R_{NE}\gamma_1 & q_2 + R_{NE}\gamma_2 \end{vmatrix} \neq 0 ,$$

$$\Delta_{34} = \begin{vmatrix} p_2 + R_L & p_1 + R_L \\ q_2 + \gamma_2 R_{FE} & q_1 + \gamma_1 R_{FE} \end{vmatrix} \neq 0$$

we obtain

$$\begin{aligned} &I_1(0,t) = A_{10}(t) + A_{11} I_3(0,t) + A_{12} I_4(0,t) \\ &I_2(0,t) = A_{20}(t) + A_{21} I_3(0,t) + A_{22} I_4(0,t) \\ &I_3(\Lambda,t) = B_{11} I_1(\Lambda,t) + B_{12} I_2(\Lambda,t) \\ &I_4(\Lambda,t) = B_{21} I_1(\Lambda,t) + B_{22} I_2(\Lambda,t) \end{aligned} .$$

Taking into account $I_k(\Lambda, t-T_k) = I_k(0,t), (k=3,4)$,

$I_k(\Lambda,t) = I_k(0, t-T_k) \; (k=1,2)$ we can rewrite the above equations in the following way:

$$\begin{aligned} &I_1(0,t) = A_{10}(t) + A_{11} I_3(\Lambda, t-T_3) + A_{12} I_4(\Lambda, t-T_4) \\ &I_2(0,t) = A_{20}(t) + A_{21} I_3(\Lambda, t-T_3) + A_{22} I_4(\Lambda, t-T_4) \end{aligned}$$

$$\begin{aligned} &I_3(\Lambda,t) = B_{11} I_1(0, t-T_1) + B_{12} I_2(0, t-T_2) \\ &I_4(\Lambda,t) = B_{21} I_1(0, t-T_1) + B_{22} I_2(0, t-T_2) \end{aligned}$$

Denoting the unknown functions by

$$I_1(0,t) \equiv I_1(t), I_2(0,t) \equiv I_2(t), I_3(t) \equiv I_3(\Lambda,t), \; I_4(t) \equiv I_4(\Lambda,t)$$

and taking into account $T_1 = T_3$, $T_2 = T_4$ we obtain the following system:

$$
\begin{aligned}
I_1(t) &= A_{10}(t) + A_{11} I_3(t-T_1) + A_{12} I_4(t-T_2) \\
I_2(t) &= A_{20}(t) + A_{21} I_3(t-T_1) + A_{22} I_4(t-T_2) \\
I_3(t) &= B_{11} I_1(t-T_1) + B_{12} I_2(t-T_2) \\
I_4(t) &= B_{21} I_1(t-T_1) + B_{22} I_2(t-T_2)
\end{aligned}
$$

The initial conditions on the intervals $[-T_1, 0], [-T_2, 0]$ can be obtained as in [12].

If $u_{10}(x), u_{20}(x), i_{10}(x), i_{20}(x)$ are periodic functions then $I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)$ are periodic functions too.

Now we formulate the main problem: to find a T_0-periodic solution of:

$$
\begin{aligned}
I_1(t) &= A_{10}(t) + A_{11} I_3(t-T_1) + A_{12} I_4(t-T_2) \\
I_2(t) &= A_{20}(t) + A_{21} I_3(t-T_1) + A_{22} I_4(t-T_2) \\
I_3(t) &= B_{11} I_1(t-T_1) + B_{12} I_2(t-T_2) \\
I_4(t) &= B_{21} I_1(t-T_1) + B_{22} I_2(t-T_2)
\end{aligned}, \qquad (9)
$$

where

$$
\begin{aligned}
&I_1(t) = I_{10}(t), t \in [-T_1, 0], \quad I_2(t) = I_{20}(t), t \in [-T_2, 0], \\
&I_3(t) = I_{30}(t), t \in [-T_1, 0], \quad I_4(t) = I_{40}(t), t \in [-T_2, 0]
\end{aligned}
$$

To prove the main theorem we use the technique of fixed point theory in uniform spaces (cf. [13]) and define suitable operators $B = (B_1, B_2, B_3, B_4)$ acting on specific function spaces.

The main result is:

Theorem 1. Let the following conditions be fulfilled:

$$
\begin{aligned}
&I_{10}(.), I_{30}(.) \in C^1_{T_0}[-T_1, 0], \quad I_{20}(.), I_{40}(.) \in C^1_{T_0}[-T_2, 0] \\
&U_S(.) \in C_{T_0}[0, \infty), \quad \breve{U}_S = \max\{|U_S(t)| : t \in [0, T_0]\}
\end{aligned} \qquad (10)
$$

Assumptions (D) and (L) are valid;

$T_1 = m_1 T_0, T_2 = m_2 T_0$ for positive integers m_1, m_2

$$
\frac{|q_2 + R_{NE}\gamma_2|}{|\Delta_{12}|}\breve{U}_S = \frac{1}{|\Delta_{12}|}\left|\begin{array}{l}\lambda_2 L_{12} + (L_{22}\lambda_2 + R_{NE}) \times \\ \times \dfrac{1 - \lambda_2^2 (L_{11}C_{11} + L_{12}C_{12})}{\lambda_2^2 (L_{12}C_{11} + L_{22}C_{12})}\end{array}\right| \breve{U}_S < \min\{I_{10}, I_{20}\} \qquad (11)
$$

$$
\frac{|q_1 + R_{NE}\gamma_1|}{|\Delta_{12}|}\breve{U}_S = \frac{1}{|\Delta_{12}|}\left|\begin{array}{l}\lambda_1 L_{12} + (L_{22}\lambda_1 + R_{NE}) \times \\ \times \dfrac{1 - \lambda_1^2 (L_{11}C_{11} + L_{12}C_{12})}{\lambda_1^2 (L_{12}C_{11} + L_{22}C_{12})}\end{array}\right| \breve{U}_S < \min\{I_{10}, I_{20}\} \qquad (12)
$$

Then there exists a unique T_0-periodic solution of (9). The proof is based on the fixed point technique.

IV. VALIDATION OF RESULTS

Since our goal is to find

$$U_{NE} = u_2(0, t); U_{FE} = u_2(\Lambda, t)$$

we have:

$$
\begin{aligned}
u_2(0,t) &= q_1 I_1(0,t) + q_2 I_2(0,t) - q_1 I_3(0,t) - q_2 I_4(0,t) = \\
&= q_1 I_1(t) + q_2 I_2(t) - q_1 I_3(t-T_1) - q_2 I_4(t-T_2) \\
u_2(\Lambda,t) &= q_1 I_1(\Lambda,t) + q_2 I_2(\Lambda,t) - q_1 I_3(\Lambda,t) - q_2 I_4(0,t) = \\
&= q_1 I_1(t-T_1) + q_2 I_2(t-T_2) - q_1 I_3(t) - q_2 I_4(t)
\end{aligned}
$$

where $(I_1(t), I_2(t), I_3(t), I_4(t))$ is the solution obtained in Theorem 1.

We have to check the conditions of our Theorem 1 referring to the data from [9]:

$L_{11} = L_G = L_R = L_{22} = 0.8529$ μH/m; $L_m = 0.3725$ μH/m;

$L_{12} = L_{21} = L_m$; $C_{11} = C_{22} = 46{:}762$ pF/m;

$C_{12} = C_{21} = -C_m = -18.036$ pF/m;

$L_{12} C_{11} + L_{22} C_{12} = L_m (C_G + C_m) - L_R C_m = 0.3725 \times 46.762 - 0.8529 \times 18.036 = 2.036 \neq 0$;

$L_{12} C_{22} + L_{11} C_{12} = L_m (C_R + C_m) - L_G C_m = 0.3725 \times 46.762 - 0.8529 \times 18.036 = 2.036 \neq 0$;

$\Delta_C = C_{11} C_{22} - C^2_{12} = (C_G + C_m)(C_R + C_m) - C^2_m = 46.762^2 - (-18.036)^2 \approx 1861.3874 > 0$;

$\Delta_L = L_G L_R - L^2_m = L_{11} L_{22} - L^2_{12} = 0.8529^2 - 0.3725^2 \approx 0.5887 > 0$;

$\lambda_1 \approx \sqrt{0.0321157} \approx 0.1792$, $\lambda_2 \approx \sqrt{0.0284} \approx 0.1686$,

$L_{22} + \lambda_1 \lambda_2 \Delta_L C_{11} = 0.8529 + 0.1792 \times 0.1686 \times 0.5887 \times 16.762 \approx 1.6846$;

$C_{22}\Delta_L + L_{11}\sqrt{\Delta_L \Delta_C} \approx 55.7546$;

$$
\begin{aligned}
\Delta_{12} &= \frac{\sqrt{D}\left(\sqrt{\Delta_L / \Delta_C} + R_S R_{FE}\right) + \lambda_1 - \lambda_2}{L_{12}C_{11} + L_{22}C_{12}} \times \\
&\times \left(\left(C_{22}\Delta_L + L_{11}\sqrt{\Delta_L \Delta_C}\right) R_{NE} + \frac{L_{22} + \lambda_1 \lambda_2 \Delta_L C_{11}}{\lambda_1 \lambda_2} R_S\right) = \\
&= 0.427 + 24.051 R_S R_{NE} + 3.5014 R_{NE} + 19.6 R_S;
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{34} &= \frac{\sqrt{D}\left(\sqrt{\Delta_L / \Delta_C} + R_\Lambda R_{FE}\right)}{L_{12}C_{11} + L_{22}C_{12}} - \\
&- \frac{\lambda_1 - \lambda_2}{L_{12}C_{11} + L_{22}C_{12}} \times \left(\begin{array}{l}\left(C_{22}\Delta_L + L_{11}\sqrt{\Delta_L \Delta_C}\right) R_{NE} + \\ + \dfrac{L_{22} + \lambda_1 \lambda_2 \Delta_L C_{11}}{\lambda_1 \lambda_2} R_\Lambda\end{array}\right) = \\
&= -\left(\begin{array}{l}0.427 + 24.051 R_\Lambda R_{NE} + \\ + 3.5014 R_{NE} + 19.6 R_\Lambda\end{array}\right);
\end{aligned}
$$

$$\gamma_1 = \frac{1-\lambda_1^2(L_{11}C_{11}+L_{12}C_{12})}{\lambda_1^2(L_{12}C_{11}+L_{22}C_{12})} \approx -0.987979;$$

$$\gamma_2 = \frac{1-\lambda_2^2(L_{11}C_{11}+L_{12}C_{12})}{\lambda_2^2(L_{12}C_{11}+L_{22}C_{12})} \approx 1.038$$

The inequalities from the main theorem are:

$$\frac{|\lambda_1 L_{12}+(L_{22}\lambda_1+R_{NE})\gamma_1|}{\Delta_{12}}\breve{U}_S = \\ = \frac{|0.0667-0.98798(0.1528+R_{NE})|}{0.427+(24.051R_S+3.5014)R_{NE}+19.6R_S}\breve{U}_S \le \\ \le \min\{I_{10}, I_{20}\}$$

$$\frac{|\lambda_2 L_{12}+(L_{22}\lambda_2+R_{NE})\gamma_2|}{\Delta_{12}}\breve{U}_S = \\ = \frac{|0.0628+1.038(0.1437+R_{NE})|}{0.427+(24.051R_S+3.5014)R_{NE}+19.6R_S}\breve{U}_S \le . \\ \le \min\{I_{10}, I_{20}\}$$

In what follows we verify how the same data satisfy the conditions generated by the particular case under weak coupling assumptions. Indeed, the main system becomes

$$\left| \begin{array}{l} (C_G+C_m)\dfrac{\partial u_G(x,t)}{\partial t}+\dfrac{\partial i_G(x,t)}{\partial x}=0 \\ -C_m\dfrac{\partial u_G(x,t)}{\partial t}+(C_R+C_m)\dfrac{\partial u_R(x,t)}{\partial t}+\dfrac{\partial i_R(x,t)}{\partial x}=0 \\ L_G\dfrac{\partial i_G(x,t)}{\partial t}+\dfrac{\partial u_G(x,t)}{\partial x}=0 \\ L_m\dfrac{\partial i_G(x,t)}{\partial t}+L_R\dfrac{\partial i_R(x,t)}{\partial t}+\dfrac{\partial u_R(x,t)}{\partial x}=0 \end{array} \right.$$

Since $|A| = \Delta_C \Delta_L \neq 0$ and then the inverse one of A does exist and then we can find the eigenvectors

$$H^{(k)} = (\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k})^T; (k = 1,2,3,4)$$

of $(B^{-1} - \mu_k I)H^{(k)} = 0;\ \mu_k = 1/\lambda_k$.

and therefore, we have

$$u_1(x,t) = \frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}}I_1(x,t) - \\ -\frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}}I_3(x,t);$$

$$u_2(x,t) = \frac{L_{12}C_{11}+L_{22}C_{12}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}}I_1(x,t) + \sqrt{\frac{L_{22}}{C_{22}}}I_2(x,t) - \\ -\frac{L_{12}C_{11}+L_{22}C_{12}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}}I_3(x,t) - \sqrt{\frac{L_{22}}{C_{22}}}I_4(x,t);$$

$$i_1(x,t) = \frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}I_1(x,t) + \frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}I_3(x,t);$$

$$i_2(x,t) = I_1(x,t)+I_2(x,t)+I_3(x,t)+I_4(x,t).$$

If we take the specific parameters again from [9]

$L_{11} = L_G = L_R = L_{22} = 0.8529$ μH/m;

$C_{11} = C_G + C_m = C_R + C_m = C_{22} = 46.762$ pF/m

it is obvious that $L_{11}\ C_{11} - L_{22}\ C_{22} = 0$. This implies $u_1(x,t) = u_G(x,t) \equiv 0$. The contradiction obtained shows the advantages of our method.

CONCLUSION

In this paper we extend the general method from [12] to investigate a 3-conductor transmission line terminated by linear loads. We reduce the mixed problem for the hyperbolic system describing TEM propagation along the lines to functional system on the boundary. The system of functional equations can be solved by a fixed point method. This means that solution can be obtained by successive approximations in an explicit form beginning with simple initial approximation. Usually such problems are treated by numerical methods or Laplace transformation method but our approach is also applicable to nonlinear boundary conditions. We point out that previous results contain an existence of harmonic solutions. Here we prove existence-uniqueness of more general periodic solution. We have shown the advantages of our method on the examples arising from the investigations of cross-talks.

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